REAL ANALYSIS MATH 607 HOMEWORK 2

Problem 1. *Let $\mathcal{B}(\mathbb{R})$ be the Borel σ-algebra of $\mathbb{R}$. Show that:*

$$\mathcal{B}(\mathbb{R}) = \mathcal{M}(\{[p,\infty) : p \text{ rational}\}).$$

Hint: You can use the fact that $\mathcal{B}(\mathbb{R})$ is generated by the open intervals.

Problem 2 (Problem 1/Page 24.)**.** *A family of sets $\mathcal{R} \subset \mathscr{P}(X)$ is called a* ring *if it is non-empty and closed under finite unions and differences. A ring that is closed under countable unions is called a* σ-ring. *Show the following assertions.*

(1) Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.
(2) If $\mathcal{R}$ is a ring (resp. σ-ring), then $\mathcal{R}$ is an algebra (resp. σ-algebra) iff $X \in \mathcal{R}$.
(3) If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.
(4) If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

Problem 3 (Problem 5/Page 24.)**.** *Let $\mathcal{E} \subset \mathscr{P}(X)$. Show that if $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$.*

Problem 4. *Let Ω be a set. $\mathcal{E} \subset \mathcal{P}(\Omega)$ is called* an elementary system *if*

(1) $\emptyset \in \mathcal{E}$,
(2) $E, F \in \mathcal{E} \Rightarrow E \cap F \in \mathcal{E}$,
(3) $E \in \mathcal{E}$ then E^c is the finite union of disjoint elements of $\mathcal{E}$.

Let $(\Omega_j, \mathcal{M}_j)$ be measurable spaces for $j = 1, 2, \ldots, n$. Show that

$$\mathcal{E} = \left\{ \prod_{j=1}^{n} E_j : E_j \in \mathcal{M}_j,\ j = 1, 2, \ldots, n \right\},$$

is an elementary system of $\Omega = \Omega_1 \times \cdots \times \Omega_n$.

Problem 5. *Show that every σ-algebra has either finite or uncountably many elements.*

Hint: Construct a sequence of mutually disjoint sets in the σ-algebra.