REAL ANALYSIS MATH 607 HOMEWORK 3

Problem 1 (5 points)**.** *Show that if an algebra $\mathscr{A} \subset \mathscr{P}(X)$ is stable under countable disjoint unions then $\mathscr{A}$ is a σ-algebra.*

Problem 2 (10 points (Problem 11/Page 27.))**.** *Assume μ is finitely additive on a σ-algebra $\mathcal{M}$. Show the following assertions.*

(a) μ is continuous from below $\implies$ μ is σ-additive.

(b) Assume $\mu(X) < \infty$. Then μ is continuous from above $\implies$ μ is σ-additive.

Problem 3 (15 points)**.** *Let* $\liminf_{j\to\infty} E_j \overset{\text{def}}{=} \cup_{k=1}^{\infty} \cap_{n\geqslant k} E_n$ *and* $\limsup_{j\to\infty} E_j \overset{\text{def}}{=} \cap_{k=1}^{\infty} \cup_{n\geqslant k} E_n$*. For $\{E_j\}_{j\geqslant 1} \subset \mathcal{M}$, show that*

(a)

$$\mu(\liminf_{j\to\infty} E_j) \leqslant \liminf_{j\to\infty} \mu(E_j)$$

(b)

$$\mu(\limsup_{j\to\infty} E_j) \geqslant \limsup_{j\to\infty} \mu(E_j),$$

provided that $\mu(\bigcup_{j=1}^{\infty} E_j) < \infty$

Problem 4 (20 points)**.** *Suppose $(X, \mathcal{M}, \mu)$ is a measure space. We call*

$$\mathcal{N}_\mu \overset{\text{def}}{=} \{A \subset X : \exists B \in \mathcal{M} \quad A \subset B \text{ and } \mu(B) = 0\}$$

the nullset *of $(X, \mathcal{M}, \mu)$ whose elements are called* nullsets*. We will now show how to extend μ to the σ-algebra generated by $\mathcal{M}$ and the nullsets.*

(a) (10 points) Show that

$$\overline{\mathcal{M}} \overset{\text{def}}{=} \{A \cup N : A \in \mathcal{M} \text{ and } N \in \mathcal{N}_\mu\}$$

is a σ-algebra.

(b) (10 points) Show that

$$\overline{\mu} : \overline{\mathcal{M}} \to [0, \infty], \quad A \cup N \mapsto \mu(A), \text{ if } A \in \mathcal{M}, N \in \mathcal{N}_\mu$$

is well-defined and a measure.

The space $(X, \overline{\mathcal{M}}, \overline{\mu})$ is called the completion of $(X, \mathcal{M}, \mu)$.

Problem 5 (20 points)**.** *Assume that the algebra $\mathcal{A}$ generates the σ-algebra $\mathcal{M}$ and assume that μ is a finite measure on $\mathcal{M}$. Show that for any $\varepsilon > 0$ and any $A \in \mathcal{M}$ there is an $\tilde{A} \in \mathcal{A}$ so that $\mu(A \Delta \tilde{A}) < \varepsilon$.*

Problem 6 ((30 points) Halmos monotone class theorem)**.** *Let X be a nonempty set. A class $\mathscr{C} \subset \mathscr{P}(X)$ is* monotone *if*

(1) $(A_n)_{n\geqslant 1} \subset \mathscr{C}$, $A_n \subseteq A_{n+1}$ *for all* $n \geqslant 1 \implies \cup_{n=1}^{\infty} A_n \in \mathscr{C}$.
(2) $(A_n)_{n\geqslant 1} \subset \mathscr{C}$, $A_n \supseteq A_{n+1}$ *for all* $n \geqslant 1 \implies \cap_{n=1}^{\infty} A_n \in \mathscr{C}$.

(a) (5 points) Show that for any $\mathscr{C} \subset \mathscr{P}(X)$ *there exists a smallest monotone class, denoted mon*$(\mathscr{C})$*, that contains* $\mathscr{C}$*.*

(b) (10 points) Show that a monotone algebra (i.e. an algebra that is monotone) is a σ*-algebra.*

(c) (15 points) Show that if $\mathscr{A}$ *is an algebra, then mon*$(\mathscr{A}) = \mathcal{M}(\mathscr{A})$*.*

Hint: Find inspiration in the proof of Dynkin theorem.