REAL ANALYSIS MATH 607 HOMEWORK 4

Problem 1. *(5 points) If* $E \in \mathscr{L}(\mathbb{R})$ *(the Lebesgue* σ*-algebra) and* $\lambda(E) > 0$ *(*λ *is the Lebesgue measure), show that there is for any* $\alpha \in (0,1)$ *an open interval* I *such that* $\lambda(E \cap I) > \alpha\lambda(I)$.

Hint: First show that you can assume that E has finite measure and then use the definition of the Lebesgue measure as an outer measure.

Problem 2. *(15 points) Let*

$$\mathcal{A}^{(\mathbb{Q})} \stackrel{\text{def}}{=} \left\{ \bigcup_{i=1}^{n} [a_i, b_i) \cap \mathbb{Q} : \begin{array}{c} n \in \mathbb{N}, \{a_i, b_i : 1 \leqslant i \leqslant n\} \subset \mathbb{Q} \cup \{\pm\infty\} \\ \text{and } a_1 < b_1 < a_2 < \dots b_n \end{array} \right\}.$$

For $A = \bigcup_{i=1}^{n} [a_i, b_i) \cap \mathbb{Q}$ *with* $-\infty \leqslant a_1 < b_1 < a_2 < \dots b_n \leqslant \infty$ *put*

$$\mu_0(A) \stackrel{\text{def}}{=} \sum_{i=1}^{n} (b_i - a_i).$$

(a) (5 points) Show that $\mathcal{A}^{(\mathbb{Q})}$ *is an algebra on* $\mathbb{Q}$
(b) (5 points) Show that μ_0 *is a finitely additive measure on* $\mathcal{A}^{(\mathbb{Q})}$.
(c) (5 points) Show that μ_0 *is* **not** *a premeasure.*

Problem 3. *(20 points) Let* μ *be a finite measure on* $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. *Given* $A \in \mathscr{B}(\mathbb{R})$, *show that for all* $\varepsilon > 0$, *there exist* K *compact and* U *open such that* $K \subset A \subset U$, *and* $\mu(U \setminus K) < \varepsilon$.

Hint: Introduce the set

$$\mathcal{D} \stackrel{\text{def}}{=} \Big\{ A \in \mathscr{B}(\mathbb{R}) : \forall \varepsilon > 0, \exists K \text{ compact}, \exists U \text{ open}, K \subset A \subset U, \mu(U \setminus K) < \varepsilon \Big\}.$$

Problem 4. *(20 points)*

(1) (10 points) Let $A \in \mathscr{L}(\mathbb{R})$ *be bounded. Suppose there is a bounded countably infinite set* $\Gamma \subset \mathbb{R}$ *for which* $\{\gamma + A\}_{\gamma \in \Gamma}$ *is a disjoint collection. Show that* $\lambda(A) = 0$.
(2) (10 points) Show that any set $A \subset \mathbb{R}$ *with* $\lambda(A) > 0$ *contains a subset* B *that is not in* $\mathscr{L}(\mathbb{R})$.

Problem 5 (Cardinality of $\mathscr{L}(\mathbb{R})$)**.** *(20 points) Let* $\mathscr{L}(\mathbb{R})$ *be the collection of Lebesgue sets.*

(1) (5 points) Show that if there exists a Borel set C *with* $\operatorname{card}(C) = \operatorname{card}(\mathbb{R})$ *and* $\lambda(C) = 0$, *then* $\operatorname{card}(\mathscr{L}(\mathbb{R})) = \operatorname{card}(\mathscr{P}(\mathbb{R}))$
(2) (15 points) Construct a Borel set C *such that* $\operatorname{card}(C) = \operatorname{card}(\mathbb{R})$ *and* $\lambda(C) = 0$.

Hint: You might want to look at Cantor middle-third set.

Problem 6 (One-dimensional Lyapunov theorem). *(20 points) Let $(X,\mathcal{M},\mu)$ be a measure space. We say that $A\in\mathcal{M}$ is an* atom*, if it satisfies the two properties below:*

(1) $\mu(A)>0$.

(2) If $A=A_1\cup A_2$, with $A_1,A_2\in\mathcal{M}$ disjoint, then $\mu(A_1)=0$ or $\mu(A_2)=0$.

Assume now that $(X,\mathcal{M},\mu)$ is an atomless measure space, i.e. there are no atoms, with $\mu(X)=1$. Show that for all $r\in[0,1]$ there exists $A\in\mathcal{M}$ such that $\mu(A)=r$.

Hint: A solution of the problem goes along the following lines:

(a) Show that for all $A\in\mathcal{M}$ with $\mu(A)>0$ there exists $B\in\mathcal{M}$ satisfying $B\subset A$ and $0<\mu(B)<\mu(A)$.

(b) Show that for all $A\in\mathcal{M}$ with $\mu(A)>0$ and all $\varepsilon>0$ there exists a measurable set $B\subset A$ such that $0<\mu(B)<\varepsilon\mu(A)$.

(c) Show that if $A\in\mathcal{M}$, then there is a measurable subset $B\subset A$, so that $\mu(A)/3\leqslant\mu(B)\leqslant\mu(A)/2$

(d) Show that for each $0\leqslant r\leqslant 1/2$, each $A\in\mathcal{M}$, and each measurable subset $B\subset A$ with $\mu(B)\geqslant\mu(A)/2$ there is a measurable subset $C\subset B$ with $r\mu(A)/3\leqslant\mu(C)\leqslant r\mu(A)$.

(e) Show that for each $A\in\mathcal{M}$, there exists a sequence $(B_n)_{n\geqslant 1}$ of measurable disjoint subsets of A such that for all $n\geqslant 1$, $\mu(\bigcup_{i=1}^n B_i)\leqslant\mu(A)/2$ and $\mu(B_n)\geqslant\frac{1}{3}[\mu(A)/2-\mu(\bigcup_{i=1}^{n-1}B_i)]$.

(f) Show that for all $A\in\mathcal{M}$ there exists $B\subset A$ such that $\mu(B)=\mu(A)/2$

(g) Show that there exists a collection $\{A_n\}_{n\geqslant 1}$ of disjoint sets satisfying $\mu(A_n)=2^{-n}$.

(h) Show that for all $r\in[0,1]$ there exists $A\in\mathcal{M}$ such that $\mu(A)=r$.

Hint:[For the hints!]

(b), (e), (g): use recursive procedures.

(c): Observe that is it sufficient to find a set such that $\mu(A)/3\leqslant\mu(B)\leqslant 2\mu(A)/3$ and proceed by contradiction by considering the parameter $r\stackrel{\text{def}}{=}\sup\{\mu(B)\colon\ B\in\mathcal{M},\ B\subset A,\ \mu(B)\leqslant\mu(A)/3\}$.

(d): use (c)