REAL ANALYSIS MATH 607 HOMEWORK 5

Problem 1. *(30 points) Throughout this problem let $f\colon X \to Y$ be a function.*

(1) *(5 points) If $\mathcal{M}$ is a σ-algebra on X, is it true that $f(\mathcal{M}) \overset{\text{def}}{=} \{f(A)\colon A \in \mathcal{M}\}$ is a σ-algebra on Y?*
(2) *(5 points) (Pushforward or direct-image σ-algebra) If $\mathcal{M}$ is a σ-algebra on X, show that $\Sigma \overset{\text{def}}{=} \{B \subset Y\colon f^{-1}(B) \in \mathcal{M}\}$ is a σ-algebra on Y.*
(3) *(5 points) (Transport Lemma) Let $\mathscr{C} \subset \mathscr{P}(Y)$. Show that $f^{-1}(\mathcal{M}(\mathscr{C})) = \mathcal{M}(f^{-1}(\mathscr{C}))$.*
(4) *(5 points) Let $\mathcal{M}_X$ be a σ-algebra on X and $\mathscr{C} \subset \mathscr{P}(Y)$. Use the transport lemma to show that f is $(\mathcal{M}_X, \mathcal{M}(\mathscr{C}))$-mesurable if $f^{-1}(\mathscr{C}) \subset \mathcal{M}_X$.*
(5) *(5 points) (Trace σ-algebra) Show that if $X \subset Y$, $\mathcal{M}$ is a σ-algebra on Y, and f is the identity function, then $f^{-1}(\mathcal{M}) = \{X \cap A\colon A \in \mathcal{M}\}$.*
(6) *(5 points) Show that if $X = Y \times Z$, $\mathcal{M}$ is a σ-algebra on Y, and f is the canonical projection from $Y \times Z$ onto Y (i.e., $f(y,z) = y$), then $f^{-1}(\mathcal{M}) = \{A \times Z\colon A \in \mathcal{M}\}$.*

Problem 2. *(15 points) Let $f, g\colon (X, \mathcal{M}) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ be measurable and $\alpha \in \mathbb{R}$.*

(1) *(1 points) Show that $\alpha \cdot f$ is measurable.*
(2) *(2 points) Show that f^2 is measurable.*
(3) *(7 points) Show that $f + g$ is measurable.*
(4) *(5 points) Show that $f \cdot g$ is measurable.*

Hint: For (3) us the density of the rationals.

Problem 3. *(15 points)*

(1) *(5 points) For all $n \geqslant 1$, let $f_n\colon (X, \mathcal{M}) \to (\overline{\mathbb{R}}, \mathscr{B}(\overline{\mathbb{R}}))$ be measurable. Show that $\inf_n f_n$, $\sup_n f_n$ and $\limsup_n f_n$ are measurable.*
(2) *(5 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that f' is a Borel function.*
(3) *(5 points) For all $n \geqslant 1$, let $f_n\colon (X, \mathcal{M}) \to (\overline{\mathbb{R}}, \mathscr{B}(\overline{\mathbb{R}}))$ be measurable. Show that $L \overset{\text{def}}{=} \{x \in X\colon \lim_{n\to\infty} f_n(x) \in \overline{\mathbb{R}}\} \in \mathcal{M}$. What measurability property can you deduce for the map $g\colon L \to \overline{\mathbb{R}}$ defined by $g(x) = \lim_n f_n(x)$?*

Problem 4. *(20 points) Assume $(X, \mathcal{M}, \mu)$ is a complete measure space.*

(a) *(10 points) If $f\colon (X, \mathcal{M}, \mu) \to \mathbb{R}$ is measurable and $f = g$ μ-almost everywhere (i.e. $\mu(\{f \neq g\}) = 0$ and we simply write μ-a.e.), then g is also measurable.*
(b) *(10 points) If $f_n\colon (X, \mathcal{M}, \mu) \to \mathbb{R}$ is measurable for $n \in \mathbb{N}$, and $\lim_{n\to\infty} f_n = f$ μ-a.e., then f is measurable.*

Problem 5. *(Approximation of Borel functions by step functions) (20 points)*

(1) *(8 points) Let $A \in \mathscr{B}(\mathbb{R})$ with finite Lebesgue measure. Show that for all $\varepsilon > 0$ there exists a subset a finite collection of disjoint open intervals $\{I_i\}_{i=1}^k$ such that $\lambda(A \Delta \cup_{i=1}^k I_i) \leqslant \varepsilon$.*

(2) *(4 points) Let $A \in \mathscr{B}(\mathbb{R})$ with finite Lebesgue measure. Show that for all $\varepsilon > 0$ there exists a measurable subset $N_\varepsilon \subset \mathbb{R}$ with $\lambda(N_\varepsilon) \leqslant \varepsilon$ and a* step *function $\psi_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ such that $\psi_\varepsilon(x) = \mathbf{1}_A(x)$ for all $x \notin N_\varepsilon$.*

(3) *(4 points) Show that for any measurable* simple *function $\varphi \colon \mathbb{R} \to \mathbb{R}$ with finite support, i.e $\lambda(\{\varphi \neq 0\}) < \infty$ and any $\varepsilon > 0$ there exists a Borel subset N_ε with $\lambda(N_\varepsilon) \leqslant \varepsilon$ and a* step *function ψ_ε such that $\psi_\varepsilon(x) = \varphi(x)$ for all $x \notin N_\varepsilon$.*

(4) *(4 points) Show that for any measurable function $f \colon \mathbb{R} \to \overline{\mathbb{R}}$ there exists a sequence of step functions that converges a.e. to f.*

Hint: For (4) use a truncation argument to reduce the problem to simple functions with finite support.