REAL ANALYSIS MATH 607
HOMEWORK #6

Problem 1 (20 points). *Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) = 1$.*

1. *(5 points) Show that* $\mu(A\cap B) - \mu(A)\mu(B) \leqslant \min\{\mu(A)(1-\mu(B)), \mu(B)(1-\mu(A))\}$.
2. *(7 points) Show that* $\mu(A)\mu(B) - \mu(A\cap B) \leqslant \min\{\mu(A)(1-\mu(A)), \mu(B)(1-\mu(B))\}$.
3. *(8 points) Show that* $|\mu(A\cap B) - \mu(A)\mu(B)| \leqslant \sqrt{\mu(A)(1-\mu(A))\mu(B)(1-\mu(B))} \leqslant \frac{1}{4}$.

Problem 2 (15 points). *Let $f\colon X \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ and $\mathcal{M}(f) \overset{\text{def}}{=} f^{-1}(\mathscr{B}(\mathbb{R}))$ the smallest σ-algebra that makes f measurable.*

1. *(2 points) Let $h\colon \mathbb{R} \to \mathbb{R}$ be a Borel map. Show that $g \overset{\text{def}}{=} h\circ f$ is $(\mathcal{M}(f), \mathscr{B}(\mathbb{R}))$-measurable.*
2. *(13 points) Show that if $g\colon (X, \mathcal{M}(f)) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is measurable, then there exists a Borel map h such that $g = h\circ f$.*

Problem 3 (20 points). *For all $n \geqslant 1$, let $f_n\colon (X, \mathcal{M}, \mu) \to [0,\infty]$ be measurable.*

1. *(10 points) Show that* $\int_X \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$.
2. *(10 points) Let $(a_{n,m})_{n\in\mathbb{N}, m\in\mathbb{N}} \subset [0,\infty)$. Show that* $\displaystyle\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{n,m} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{n,m}$.

Problem 4 (20 points). *For all $n \geqslant 1$, let $f_n\colon (X, \mathcal{M}, \mu) \to [0,\infty]$ be measurable and such that $\{f_n\}_n$ decreases pointwise to f, and $\int_X f_1 d\mu < \infty$. Show that* $\int_X f d\mu = \displaystyle\lim_{n\to\infty} \int_X f_n d\mu$.

Problem 5 (25 points). *Let $-\infty < a < b < \infty$ and $f\colon [a,b] \to \mathbb{R}$. Recall that f is Riemann integrable if and only if there exist step functions $\{\phi_n\}_{n\geqslant 1}$ and $\{\psi_n\}_{n\geqslant 1}$ on $[a,b]$ such that*

1. $\forall n \geqslant 1, |\phi_n - f| \leqslant \psi_n$,
2. $\lim_{n\to\infty} \int_a^b \psi_n(x)dx = 0$,

and the Riemann integral of f is given by $\int_a^b f(x)dx \overset{\text{def}}{=} \displaystyle\lim_{n\to\infty}\int_a^b \phi_n(x)dx$.

1. *(10 points) Show that if $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable, then f is Lebesgue measurable and that $\int_a^b f(x)dx = \int_{[a,b]} f d\lambda$, where λ is the Lebesgue measure.*
2. *(15 points) Show that*

$$\lim_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)^n e^{\alpha x}dx = \int_{[0,\infty)} e^{(\alpha-1)x} d\lambda(x).$$

Hint: For (1), you might want to introduce $\alpha_n \overset{\text{def}}{=} \phi_n - \psi_n$, $\beta_n \overset{\text{def}}{=} \phi_n + \psi_n$, $\tilde{\alpha}_n \overset{\text{def}}{=} \max\{\alpha_1, \ldots, \alpha_n\}$, and $\tilde{\beta}_n \overset{\text{def}}{=} \min\{\beta_1, \ldots, \beta_n\}$.