REAL ANALYSIS MATH 607
HOMEWORK #7

Problem 1 (20 points)**.**

(1) (5 points) Let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$*. Show that* $\liminf_n(-x_n) = -\limsup_n x_n$*.*

(2) (5 points) Show that if for all $n \geqslant 1$*,* $f_n\colon (X, \mathcal{M}, \mu) \to \overline{\mathbb{R}}$ *are measurable functions such that* $f_n \geqslant 0$ *μ-a.e., then*

$$\int_X \liminf_n f_n d\mu \leqslant \liminf_n \int_X f_n d\mu.$$

(3) (5 points) (Scheffé's Theorem) Let $(f_n)_n \subset \mathscr{L}^1_{\mathbb{R}}(X, \mathcal{M}, \mu)$ *that converges μ-a.e. to* $f \in \mathscr{L}^1_{\mathbb{R}}(X, \mathcal{M}, \mu)$ *and such* $\lim_n \int_X f_n d\mu = \int_X f d\mu$*. Show that if for all* $n \geqslant 1$*,* $f_n \geqslant 0$ *μ-a.e. then* $\lim_n \int_X |f_n - f| d\mu = 0$*.*

(4) (5 points) Does Scheffé's Theorem remain true if the functions are not nonnegative μ-a.e.?

Problem 2 (15 points)**.** *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be Lebesgue integrable and* λ *denote the Lebesgue measure (i.e.* $f \in \mathscr{L}^1(\lambda)$*).*

(1) (5 points) Show that for all $a \in \mathbb{R}$*, the map* $\tau_a(f)\colon \mathbb{R} \to \mathbb{R}$*, defined by* $x \mapsto f(x+a)$ *is Lebesgue measurable and* $\int_{\mathbb{R}} \tau_a(f) d\lambda = \int_{\mathbb{R}} f d\lambda$*.*

(2) (5 points) Show that for any $A \in \mathscr{L}(\mathbb{R})$ *and* $c \neq 0$*,* $\lambda(cA) = |c|\lambda(A)$*.*

(3) (5 points) Show that for all $c \neq 0$*, the map* $\delta_c(f)\colon \mathbb{R} \to \mathbb{R}$*, defined by* $x \mapsto f(cx)$ *is Lebesgue measurable and* $\int_{\mathbb{R}} \delta_c(f) d\lambda = \frac{1}{|c|} \int_{\mathbb{R}} f d\lambda$*.*

Problem 3 (20 points)**.**

(1) (10 points) Show the following version of the Dominated Convergence Theorem. For all $n \geqslant 1$*, let* $f_n\colon (X, \mathcal{M}, \mu) \to \mathbb{R}$ *be measurable functions satisfying :*

(a) $(f_n)_n$ *converges pointwise μ-a.e..*

(b) There exists a map $g \in \mathscr{L}^1_{\overline{\mathbb{R}}_+}(X, \mathcal{M}, \mu)$ *such that for all* $n \geqslant 1$*,* $|f_n| \leqslant g$ *μ-a.e..*

Then, there exists $f \in \mathscr{L}^1_{\mathbb{R}}(X, \mathcal{M}, \mu)$ *such that*

(i) $f_n \overset{p.w.}{\to} f$ *μ-a.e..*

(ii) $\lim_n \int_X |f_n - f| d\mu = 0$

(2) (5 points) Show that if $\lim_n \int_X |f_n - f| d\mu = 0$ *then* $\lim_n \int_X f_n d\mu = \int_X f d\mu$*.*

(3) (5 points) Does the DCT remain true if (b) *is satisfied but not* (a)*?*

Hint: You can use the version of the DCT proved in class.

Problem 4 (25 points)**.** *Let* $f\colon (X, \mathcal{M}, \mu) \to \overline{\mathbb{R}}_+$ *measurable.*

(1) (2 points) Show that for all $C > 0$*,*

$$\mu(\{f \geqslant C\}) \leqslant \frac{1}{C} \int_X f d\mu.$$

(2) (5 points) Show that if $\int_X f d\mu < \infty$*, then* $\mu(\{f = \infty\}) = 0$

(3) *(3 points) For all $n \geqslant 1$, let $\varphi_n \colon (X, \mathcal{M}, \mu) \to \overline{\mathbb{R}}$ be measurable functions such that $\sum_{n=1}^{\infty} \int_X |\varphi_n| d\mu < \infty$. Show that $\sum_{n=1}^{\infty} \varphi_n$ is well-defined μ-a.e. and that $\int_X (\sum_{n=1}^{\infty} \varphi_n) d\mu = \sum_{n=1}^{\infty} \int_X \varphi_n d\mu$.*

(4) *(5 points) (Borel-Cantelli Lemma) Let $(A_n)_{n \geqslant 1} \subset \mathcal{M}$. Show that if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(\limsup_n A_n) = 0$.*

(5) *(5 points) Let $(A_n)_{n \geqslant 1} \subset \mathcal{M}$ such that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ and $f \in \mathscr{L}^1_{\mathbb{R}}(\mu)$. Show that $\lim_n \int_{A_n} |f| d\mu = 0$.*

(6) *(5 points) ("Continuity" of the integral with respect to the measure) Let $f \in \mathscr{L}^1_{\mathbb{R}}(\mu)$. Show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $A \in \mathcal{M}$, if $\mu(A) \leqslant \delta$, then $\int_A |f| d\mu \leqslant \varepsilon$.*

Problem 5 (20 points)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space with finite measure (i.e. $\mu(X) < \infty$) and $f_n \colon (X, \mathcal{M}, \mu) \to \mathbb{R}$, $n \geqslant 1$, be measurable functions.*

(1) *(2 points) Show that the set of point where $(f_n)_n$ converges is*

$$C \stackrel{\text{def}}{=} \bigcap_{k \geqslant 1} \bigcup_{n \geqslant 1} \bigcap_{i,j \geqslant n} \{|f_i - f_j| \leqslant \frac{1}{k}\}.$$

(2) *(5 points) Assume that $(f_n)_n$ converges μ-a.e. to f, and for all $n, k \geqslant 1$ let*

$$A_n^k \stackrel{\text{def}}{=} \bigcup_{r=1}^{n} \bigcap_{i=r}^{\infty} \{|f_i - f| \leqslant \frac{1}{k}\}.$$

Show that for all $\varepsilon > 0$ and for all $k \geqslant 1$, there exists $n_{k,\varepsilon} \geqslant 1$ such that $\mu((A^k_{n_{k,\varepsilon}})^c) < \frac{\varepsilon}{2^k}$.

(3) *(5 points) (Egoroff's Theorem) Show that for all $\varepsilon > 0$, there exists $A \in \mathcal{M}$ such that $\mu(A^c) < \varepsilon$ and $(f_n)_n$ converges uniformly on A to f.*

(4) *(3 points) Does Egoroff's Theorem remain true if $\mu(X) = \infty$?*

(5) *(5 points) Assuming that $\mu(X) < \infty$, show that if $(f_n)_n$ converges μ-a.e. then $(f_n)_n$ converges in measure.*