REAL ANALYSIS MATH 607
HOMEWORK #8

Problem 1 (30 points)**.** *Recall that* $(\mathsf{X},\mathsf{d}_\mathsf{X})$ *is a semi-metric space if* $\mathsf{d}_\mathsf{X}\colon \mathsf{X}\times\mathsf{X}\to[0,\infty)$ *satisfies for all* $x,y,z\in\mathsf{X}$*:*

(1) $\mathsf{d}_\mathsf{X}(x,x)=0$,
(2) $\mathsf{d}_\mathsf{X}(x,y)=\mathsf{d}_\mathsf{X}(y,x)$,
(3) $\mathsf{d}_\mathsf{X}(x,y)\leqslant\mathsf{d}_\mathsf{X}(x,z)+\mathsf{d}_\mathsf{X}(z,y)$,

and d_X *is a metric if moreover* $\mathsf{d}_\mathsf{X}(x,y)=0\implies x=y$.

Let $\mathscr{L}_1(X,\mathcal{M},\mu)$ *be the set of all integrable functions* $f\colon(X,\mathcal{M},\mu)\to\mathbb{R}$.

(1) *(3 points) Show that* $\mathsf{d}_1(f,g)\overset{\text{def}}{=}\int_X|f-g|d\mu$ *is a semi-metric on* $\mathscr{L}_1(X,\mathcal{M},\mu)$.
(2) *(3 points) Show that the relation* $f\sim g\iff f=g$ μ*-a.e. is an equivalence relation on* $\mathscr{L}_1(X,\mathcal{M},\mu)$.
(3) *(2 points) Show that if* $L_1(X,\mathcal{M},\mu)\overset{\text{def}}{=}\mathscr{L}_1(X,\mathcal{M},\mu)_{/\sim}=\{\bar f\colon f\in\mathscr{L}_1(X,\mathcal{M},\mu)\}$ *is the set of equivalence classes of integrable functions for the relation* $\sim$*, then* $\bar{\mathsf{d}}_1(\bar f,\bar g)\overset{\text{def}}{=}\int_X|f-g|d\mu$ *is a metric on* $L_1(X,\mathcal{M},\mu)$.
(4) *(5 points) Assume that* $\mu(X)<\infty$ *and consider the function* $\mathsf{d}_m(f,g)\overset{\text{def}}{=}\int_X\min\{1,|f-g|\}d\mu$*. Show that* d_m *is a metric on* $L_0(X,\mathcal{M},\mu)$ *the set of equivalence classes of measurable functions (for the* μ*-a.e. equivalence relation* $\sim$*).*
(5) *(10 points) Let* $f,f_n,n\geqslant1$ *be real-valued measurable functions. Show that* $\lim_{n\to\infty}\mathsf{d}_m(f,f_n)=0\iff(f_n)_n$ *converges to* f *in* μ*-measure.*
(6) *(4 points) Show that* d_m *is a complete metric.*
(7) *(3 points) Let* $f,f_n,n\geqslant1$ *be real-valued measurable functions. Is there a metric* d *on* $L_0(X,\mathcal{M},\mu)$ *such that* $\lim_{n\to\infty}\mathsf{d}(f,f_n)=0\iff(f_n)_n$ *converges to* f μ*-a.e.?*

Hint: For (7) you can use the fact (and prove it if you feel like) that in any metric space $(\mathsf{X},\mathsf{d}_\mathsf{X})$, a sequence $(x_n)_n\subset\mathsf{X}$ converges to x if and only if every subsequence of $(x_n)_n$ has a further subsequence that converges to x.

Problem 2 (25 points)**.** *(Existence and uniqueness of the product measure) Let* $(X,\mathcal{M},\mu)$ *and* $(Y,\mathcal{N},\nu)$ *be two* σ*-finite measure spaces.*

(1) *(5 points) Show that if two measures* m_1 *and* m_2 *on the product* σ*-algebra* $\mathcal{M}\otimes\mathcal{N}$ *satisfy for all* $A\in\mathcal{M}$*, and all* $B\in\mathcal{N}$*,* $m_1(A\times B)=\mu(A)\nu(B)=m_2(A\times B)$*, then necessarily* $m_1=m_2$ *on* $\mathcal{M}\otimes\mathcal{N}$.
(2) *(10 points) Show that for all* $C\in\mathcal{M}\otimes\mathcal{N}$*, the maps* $x\in X\mapsto\nu(C_x)$ *and* $y\in Y\mapsto\mu(C^y)$ *are well-defined and measurable.*
(3) *(5 points) Show that the maps* $m_1,m_2\colon\mathcal{M}\otimes\mathcal{N}\to[0,\infty]$ *defined by*

$$m_1(C)\overset{\text{def}}{=}\int_X\nu(C_x)d\mu(x)\quad\text{and}\quad m_2(C)\overset{\text{def}}{=}\int_Y\mu(C^y)d\nu(y),$$

are well-defined and are measures on $\mathcal{M}\otimes\mathcal{N}$.
(4) *(5 points) Show that if* $C=A\times B$ *with* $A\in\mathcal{M}$ *and* $B\in\mathcal{N}$ *then* $m_1(A\times B)=m_2(A\times B)=\mu(A)\nu(B)$ *and conclude.*

Problem 3 (25 points)**.** *For all $n \geqslant 1$, let $(X_n, \mathcal{M}_n, \mu_n)$ be a probability space, i.e. a measure space such that $\mu_n(X_n) = 1$.*

(1) (5 points) Show that $\mathcal{A} \overset{\text{def}}{=} \left\{A \times \prod_{k=n+1}^{\infty} X_k : A \in \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n, n \in \mathbb{N}\right\}$ is an algebra on $\prod_{n=1}^{\infty} X_n$.

(2) (10 points) Show that the map $\nu\colon \mathcal{A} \to [0,1]$ given by

$$\nu(A \times \prod_{k=n+1}^{\infty} X_k) \overset{\text{def}}{=} (\mu_1 \otimes \cdots \otimes \mu_n)(A),$$

is well-defined and a premeasure on $\mathcal{A}$ of total measure 1.

(3) (10 points) The product σ-algebra on $\prod_{n=1}^{\infty} X_n$ is denoted by $\otimes_{n=1}^{\infty} \mathcal{M}_n$, and is defined as the σ-algebra generated by $\mathcal{A}$. Show that there exists a unique probability measure m on $\otimes_{n=1}^{\infty} \mathcal{M}_n$ such that $m(A_1 \times \cdots \times A_n \times \prod_{k=n+1}^{\infty} X_k) = \mu_1(A_1) \cdots \mu_n(A_n)$ for all $n \geqslant 1$ and $A_n \in \mathcal{M}_n$.

Problem 4 (20 points)**.** *(Layer-cake/Distribution formula) Let μ be a σ-finite measure on $(X, \mathcal{M})$ and $f\colon (X, \mathcal{M}) \to [0, \infty)$ be measurable. Show that*

$$\int_X f d\mu = \int_0^{\infty} \mu(\{f > t\}) dt$$