REAL ANALYSIS MATH 607
HOMEWORK #9

Problem 1 (25 points). *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\mathcal{N}_\mu \stackrel{\text{def}}{=} \{N \subset X \colon \exists A \in \mathcal{M}, N \subset A, \mu(A) = 0\}$ be the null set of μ. Recall that the completion of $\mathcal{M}$ is $\overline{\mathcal{M}}^\mu = \mathcal{M} \cup \mathcal{N}_\mu$, and the completion of μ is $\bar{\mu}$ defined on $\overline{\mathcal{M}}^\mu$ by $\bar{\mu}(A \cup N) = \mu(A)$. The complete measure space $(X, \overline{\mathcal{M}}^\mu, \bar{\mu})$ is called the completion of $(X, \mathcal{M}, \mu)$.*

(1) *(5 points) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces. Show that*

$$(\mathscr{P}(X) \times \mathcal{N}_\nu) \cup (\mathcal{N}_\mu \times \mathscr{P}(Y)) \subset \mathcal{N}_{\mu \otimes \nu}.$$

(2) *(5 points) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces. Show that if $\mathcal{A} \neq \mathscr{P}(X)$ and $\mathcal{N}_\nu \neq \{\emptyset\}$, or $\mathcal{B} \neq \mathscr{P}(Y)$ and $\mathcal{N}_\mu \neq \{\emptyset\}$ then $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$ is not complete. What can you deduce for the measure space $(\mathbb{R}^2, \mathscr{L}(\mathbb{R}) \otimes \mathscr{L}(\mathbb{R}), \bar{\lambda} \otimes \bar{\lambda})$ where $\bar{\lambda}$ is the completion of the Lebesgue measure on the Lebesgue σ-algebra $\mathscr{L}(\mathbb{R}) \stackrel{\text{def}}{=} \overline{\mathscr{B}(\mathbb{R})}^\lambda$?*

(3) *(10 points) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces. Show that*

$$\mathcal{N}_{\bar{\mu} \otimes \bar{\nu}} = \mathcal{N}_{\mu \otimes \nu}, \qquad \overline{\overline{\mathcal{A}}^\mu \otimes \overline{\mathcal{B}}^\nu}^{\bar{\mu} \otimes \bar{\nu}} = \overline{\mathcal{A} \otimes \mathcal{B}}^{\mu \otimes \nu}, \qquad \overline{\bar{\mu} \otimes \bar{\nu}} = \overline{\mu \otimes \nu}.$$

(4) *(5 points) Show that $\overline{\lambda_2} = \overline{\bar{\lambda} \otimes \bar{\lambda}}$, and $\overline{\mathscr{L}(\mathbb{R}) \otimes \mathscr{L}(\mathbb{R})}^{\bar{\lambda} \otimes \bar{\lambda}} = \mathscr{L}(\mathbb{R}^2)$, where λ_2 is the 2-dimensional Lebesgue measure.*

Problem 2 (15 points). *(Old qualifier exam problem) Suppose that for $n \in \mathbb{N}$, $E_n \in \mathscr{B}(\mathbb{R})$ is a Borel set. Assume that $f(x) = \lim_{n \to \infty} \mathbf{1}_{E_n}(x)$ exists for λ-almost all $x \in \mathbb{R}$.*

(1) *(5 points) Show that f is λ-a.e. equal to a characteristic function of a Borel set $E \subset \mathbb{R}$.*

(2) *Show that for any $g \in \mathscr{L}^1$:*

$$\int_E g \, d\lambda = \lim_{n \to \infty} \int_{E_n} g \, d\lambda.$$

(3) *Let $E_n, n \geqslant 1$, and E be Borel sets. Establish a necessary and sufficient condition for $\mathbf{1}_{E_n} \to \mathbf{1}_E$ in $\mathscr{L}^1$, i.e. $\int_{\mathbb{R}} |\mathbf{1}_{E_n} - \mathbf{1}_E| d\lambda \to 0$.*

Problem 3 (20 points). *(Old qualifier exam problem) Let $f \colon [0,1] \to \mathbb{R}$ be integrable (with respect to Lebesgue measure λ) and nonnegative. Define*

$$G_f \stackrel{\text{def}}{=} \{(x, y) \colon 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant f(x)\}.$$

Show that G_f is measurable in $[0,1] \times \mathbb{R}$ and that

$$\lambda_2(G_f) = \int_{[0,1]} f \, d\lambda.$$

Problem 4 (15 points)**.** *(Old qualifier exam problem) Let* $f\colon (0,1) \to \mathbb{R}$ *be integrable with respect to the Lebesgue measure (denoted dx) on* $(0,1)$*. For* $0 < x < 1$ *define*

$$g(x) = \int_x^1 t^{-1} f(t) dt.$$

Prove that g is Lebesgue integrable on $(0,1)$ *and that*

$$\int_0^1 g(x) dx = \int_0^1 f(x) dx.$$

Problem 5 (25 points)**.** *Let* $\mu\colon (X, \mathcal{M}) \to \overline{\mathbb{R}}$ *be a signed measure.*

(1) (3 points) Let $A, B \in \mathcal{M}$*. Show that if* $A \subset B$ *and* $\mu(B) \in \mathbb{R}$ *then* $\mu(A) \in \mathbb{R}$ *and* $\mu(B \setminus A) = \mu(B) - \mu(A)$*.*

(2) (5 points) Show that if $\mu(\cup_{n=1}^{\infty} A_n) \in \mathbb{R}$ *with* $(A_n)_n$ *a disjoint sequence in* $\mathcal{M}$ *then* $\sum_{n=1}^{\infty} |\mu(A_n)| < \infty$

(3) (5 points) Let $f\colon (X, \mathcal{M}) \to \overline{\mathbb{R}}$ *be measurable and* μ *a positive measure on* $(X, \mathcal{M})$ *such that* $\int_X f^+ d\mu < \infty$*. Show that* $\mu_f(A) \overset{\text{def}}{=} \int_A f d\mu$ *defines a signed measure on* $(X, \mathcal{M})$*.*

(4) (2 points) Give a simple example of a finite signed measure such that monotonicity and subadditivity fail.

(5) For concreteness, assume that μ *does not take the value* $-\infty$*. For this question you are not allowed to use the existence of a Hahn decomposition as we will give an alternate proof of the Hahn decomposition theorem.*

(a) (5 points) Let $A \in \mathcal{M}$ *such that* $\mu(A) \leqslant 0$*. Show that there exists a measurable subset* $N \subset A$ *that is negative for* μ *and such that* $\mu(N) \leqslant \mu(A)$*.*

(b) (5 points) Show the existence of a Hahn decomposition for μ*.*

Hint: For (5a) Try to build recursively a disjoint sequence $(B_n)_n$ of subsets of A of "largest possible measure" and consider the set $A \setminus \cup_n B_n$.

For (5b) Consider the number $s_0 \overset{\text{def}}{=} \inf\{\mu(A) \colon A \subset X, A \in \mathcal{M}\}$ and construct, using (5a), a negative set A_1 such that $\mu(A_1) \leqslant \max\{s_0/2, -1\}$. Continue recursively to construct disjoint negative sets $(A_n)_n$ and show that $\cup_n A_n$ is negative and its complement is positive.