Homework 1

5.3: # 8, 14, 24, 34, 60, 62, 70, 75

5.5: # 8, 34, 70, 90

$\boxed{5.3}$

8. $g(x) = \int_1^x \ln(1+t^2)\, dt$

Apply the Fundamental Theorem of Calculus, Part 1 (FTC 1)

Here, $f(t) = \ln(1+t^2)$

$g'(x) = f(x)$

$\boxed{g'(x) = \ln(1+x^2)}$

14. $h(x) = \int_1^{\sqrt{x}} \frac{z^2}{z^4+1}\, dz$

We can apply the Chain Rule and FTC 1

Reminder:

If $h(x) = f(g(x))$

$h'(x) = f'(g(x)) \cdot g'(x)$

Here

$g(x) = \sqrt{x}$

$f(x) = \int_1^x \frac{z^2}{z^4+1}\, dz$

Then,

$$h'(x) = \frac{(\sqrt{x})^2}{(\sqrt{x})^4 + 1} \cdot (\sqrt{x})'$$

$$= \frac{x}{x^2+1} \cdot \frac{1}{2\sqrt{x}} = \boxed{\frac{\sqrt{x}}{2(x^2+1)}}$$

24. $\int_1^8 x^{-2/3}\,dx = \frac{x^{1/3}}{1/3}\Big|_1^8 = 3x^{1/3}\Big|_1^8$

$= 3(8^{1/3} - 1^{1/3}) = 3(2-1) = \boxed{3}$

34. $\int_0^3 (2\sin x - e^x)\,dx$

$= 2\int_0^3 \sin x\,dx - \int_0^3 e^x\,dx$

$= -2\cos x - e^x \Big|_0^3$

$= -(2\cos x + e^x)\Big|_0^3$

$= -[2\cos(3) + e^3 - (2\cos(0) + e^0)]$

$= -2\cos(3) - e^3 + 2 + 1$

$= \boxed{-2\cos(3) - e^3 + 3}$

60. $g(x) = \int_{1-2x}^{1+2x} t\sin t\,dt$

$= \int_{1-2x}^{0} t\sin t\,dt + \int_0^{1+2x} t\sin t\,dt$

$= -\int_0^{1-2x} t\sin t\,dt + \int_0^{1+2x} t\sin t\,dt$

Apply FTC 1 & the chain Rule to take the derivative

$g'(x) = -(1-2x)\sin(1-2x)\cdot(1-2x)'$
$\quad + (1+2x)\sin(1+2x)\cdot(1+2x)'$

$= -(1-2x)\sin(1-2x)(-2)$
$\quad + (1+2x)\sin(1+2x)\cdot(2)$

$= \boxed{2(1-2x)\sin(1-2x) + 2(1+2x)\sin(1+2x)}$

62. $F(x) = \int_{\sqrt{x}}^{2x} \arctan t \, dt$

$= \int_{\sqrt{x}}^{0} \arctan t \, dt + \int_{0}^{2x} \arctan t \, dt$

$= -\int_{0}^{\sqrt{x}} \arctan t \, dt + \int_{0}^{2x} \arctan t \, dt$

$F'(x) = -\arctan(\sqrt{x}) \cdot (\sqrt{x})'$
$+ \arctan(2x) \cdot (2x)'$
$= -\arctan(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} + \arctan(2x) \cdot (2)$
$= \boxed{\frac{-\arctan(\sqrt{x})}{2\sqrt{x}} + 2\arctan(2x)}$

70. $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt$

a) First, notice that

(*) $\int_0^x e^{-t^2} \, dt = \frac{1}{2}\sqrt{\pi} \operatorname{erf}(x)$

From property 5 of definite integrals (found in section 5.2)

$\int_a^b e^{-t^2} \, dt = \int_a^0 e^{-t^2} \, dt + \int_0^b e^{-t^2} \, dt$

$= -\int_0^a e^{-t^2} \, dt + \int_0^b e^{-t^2} \, dt$

Now, I will apply (*)

$= -\frac{1}{2}\sqrt{\pi} \operatorname{erf}(a) + \frac{1}{2}\sqrt{\pi} \operatorname{erf}(b)$

$= \frac{1}{2}\sqrt{\pi} \left[\operatorname{erf}(b) - \operatorname{erf}(a)\right]$.

b) $y = e^{x^2} \operatorname{erf}(x)$

I will apply the product rule.

First, I will compute:

$(e^{x^2})' = e^{x^2} \cdot 2x = 2xe^{x^2}$ (chain rule)

$(\operatorname{erf}(x))' = \frac{d}{dx}\left(\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt\right)$

$= \frac{2}{\sqrt{\pi}} e^{-x^2}$ (FTC1)

Now,

$y' = (e^{x^2})' \cdot \operatorname{erf}(x) + e^{x^2} \cdot (\operatorname{erf}(x))'$

$= 2xe^{x^2} \cdot \operatorname{erf}(x) + e^{x^2} \cdot \frac{2}{\sqrt{\pi}} e^{-x^2}$

$= 2xy + \frac{2}{\pi}$

75. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i^4}{n^5} + \frac{i}{n^2}\right)$ on $[0, 1]$

Recall from Section 5.2

Theorem 4: If f is integrable on $[a, b]$, then

(*) $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

where $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$

Here,

$\Delta x = \frac{1-0}{n} = \frac{1}{n}$, $x_i = 0 + i\left(\frac{1}{n}\right)$

$x_i = \frac{i}{n}$

First, I will factor out $\Delta x = \frac{1}{n}$ to get the equation in the proper form.

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{i^4}{n^4} + \frac{i}{n}\right)$$

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{1}{n}\left(\left(\frac{i}{n}\right)^4 + \frac{i}{n}\right)$$

I will now substitute $\Delta x = \frac{1}{n}$, $x_i = \frac{i}{n}$

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} \Delta x\left(x_i^4 + x_i\right)$$

Now, I can identify $f(x)$ from (*).

$f(x) = x^4 + x$

Therefore

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i^4}{n^5} + \frac{i}{n^2}\right) = \int_0^1 (x^4 + x)\,dx$$

Now evaluate the integral.

$$\int_0^1 (x^4 + x)\,dx = \frac{x^5}{5} + \frac{x^2}{2}\Big|_0^1$$

$$= \frac{(1)^5}{5} + \frac{(1)^2}{2} - \left(\cancel{\frac{(0)^5}{5} + \frac{(0)^2}{2}}^{\,0}\right) = \frac{1}{5} + \frac{1}{2} = \boxed{\frac{7}{10}}$$

$\boxed{5.5}$

8. $\int x^2 e^{x^3}\, dx$

Let $\underbrace{u = x^3}$

$\frac{du}{dx} = 3x^2$

$du = 3x^2\, dx$

$\underbrace{dx = \frac{du}{3x^2}}$

substitute:

$\int \cancel{x^2} e^u \left(\frac{du}{3\cancel{x^2}}\right) = \frac{1}{3}\int e^u\, du$

$= \frac{1}{3} e^u + C = \boxed{\frac{1}{3} e^{x^3} + C}$

34. $\int \frac{\cos(\pi/x)}{x^2}\, dx$

Let $u = \frac{\pi}{x} = \pi x^{-1}$

$\frac{du}{dx} = -\pi x^{-2}$

$du = -\pi x^{-2}\, dx$

$dx = -\frac{x^2}{\pi}\, du$

Substitute:

$\int \frac{\cos(u)}{\cancel{x^2}} \left(-\frac{\cancel{x^2}}{\pi}\, du\right) = -\frac{1}{\pi}\int \cos(u)\, du$

$= -\frac{1}{\pi}\sin(u) + C = \boxed{-\frac{1}{\pi}\sin(\pi/x) + C}$

70. $\int_0^2 (x-1)\, e^{(x-1)^2}\, dx$

$u = (x-1)^2$

$\frac{du}{dx} = 2(x-1)$

$du = 2(x-1)\,dx$

$dx = \frac{du}{2(x-1)}$

Evaluate limits of integration in terms of u:

@ $x = 0$, $u = (0-1)^2 = 1$

@ $x = 2$, $u = (2-1)^2 = 1$

Substitute:

$\int_1^1 (\cancel{x-1}) e^u \left(\frac{du}{2(\cancel{x-1})} \right) = \boxed{0}$

Since the limits are equal, the definite integral is <u>zero</u>.

90. If f is continuous on $\mathbb{R}$, prove that $\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx$

<u>Proof</u>:

Let $u = x + c$

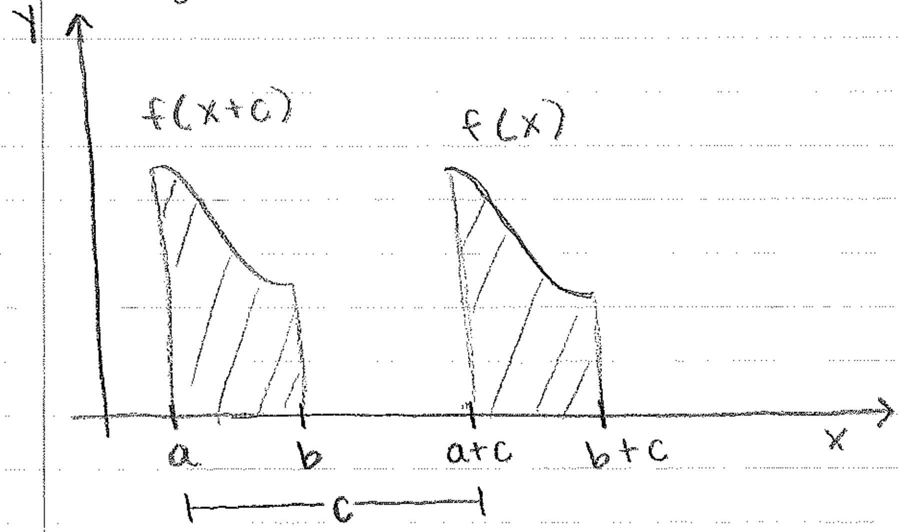
Then, $du = dx$, and the limits in terms of u are $u = a + c$ when $x = a$, $u = b + c$ when $x = b$.

Therefore,

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(u)\,du = \int_{a+c}^{b+c} f(x)\,dx.$$

∎

Diagram:



The diagram shows f(x+c) is a translation of f(x) by the distance c.