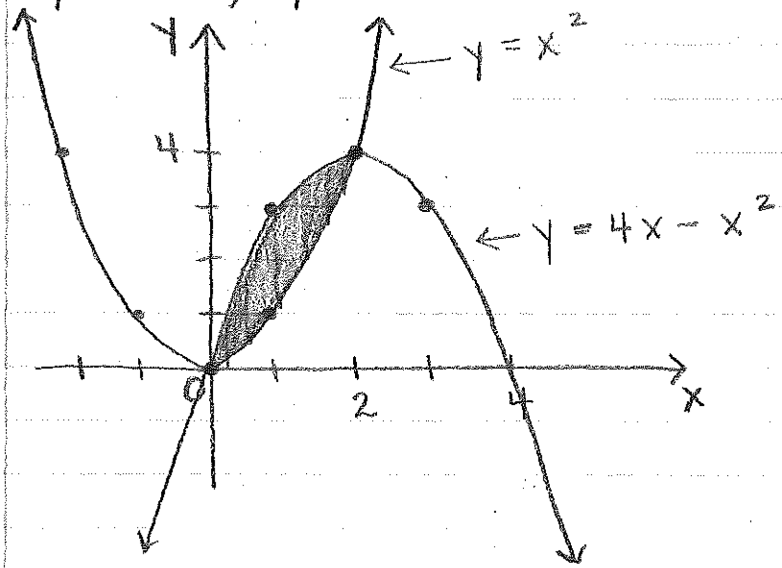
MATH 172 Homework 2

6.1 # 14, 16, 22

6.2 # 2, 6, 14, 16, 48

$\boxed{6.1}$

14. $y = x^2$, $y = 4x - x^2$



Find points of intersection:

$x^2 = 4x - x^2$

$2x^2 - 4x = 0$

$2x(x-2) = 0$

$2x = 0 \quad x - 2 = 0$

$\Rightarrow \quad x = 0, 2$

We see that $y = 4x - x^2$ is above $y = x^2$ in the shaded region

$\Rightarrow$

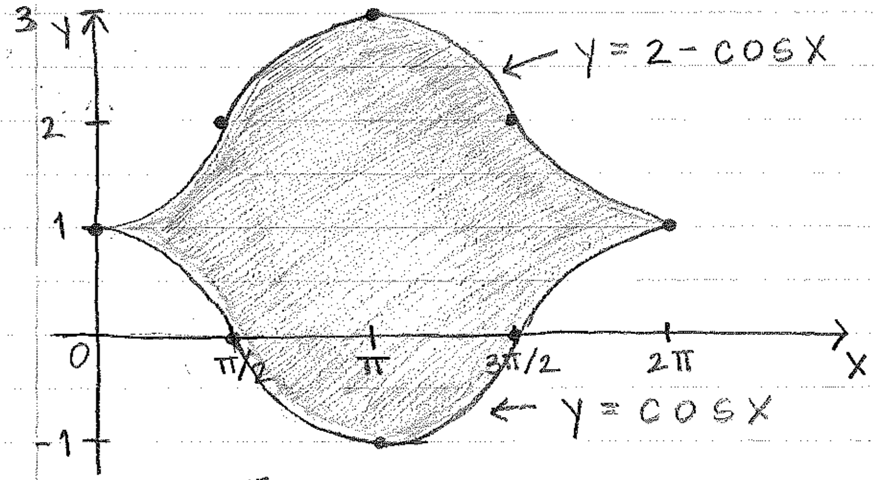
$$A = \int_0^2 (4x - x^2 - x^2)\,dx$$

$$= \int_0^2 (4x - 2x^2)\,dx$$

$$= 2x^2 - \tfrac{2}{3}x^3 \Big|_0^2$$

$$= 2(2)^2 - \tfrac{2}{3}(2)^3 - (0) = 8 - 16/3 = \boxed{8/3}$$

16. $y = \cos x$, $y = 2 - \cos x$, $0 \le x \le 2\pi$



$$A = \int_0^{2\pi} (2 - \cos x - \cos x)\, dx$$

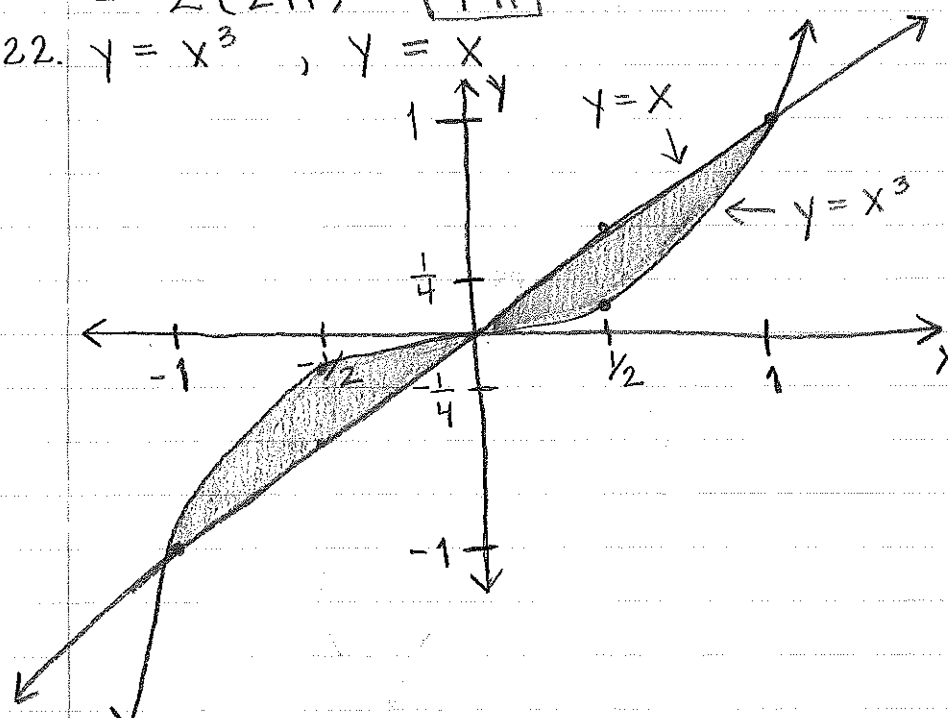
$$= \int_0^{2\pi} (2 - 2\cos x)\, dx = 2\int_0^{2\pi} (1 - \cos x)\, dx$$

$$= 2\Big[x - \sin x \Big]_0^{2\pi}$$

$$= 2\left[2\pi - \sin(2\pi) - (0 - \sin 0)\right]$$

$$= 2(2\pi) = \boxed{4\pi}$$

22. $y = x^3$, $y = x$



Find points of intersection:

$x^3 = x$

$x^3 - x = 0$

$x(x^2-1) = 0$

$x = 0 \; , \; x^2 - 1 = 0$

$\Rightarrow \; x = 0, 1, -1.$

$$A = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx$$

$$= 2\int_{0}^{1} (x - x^3)\,dx$$

since the functions are odd, the 2 shaded regions have equal area.

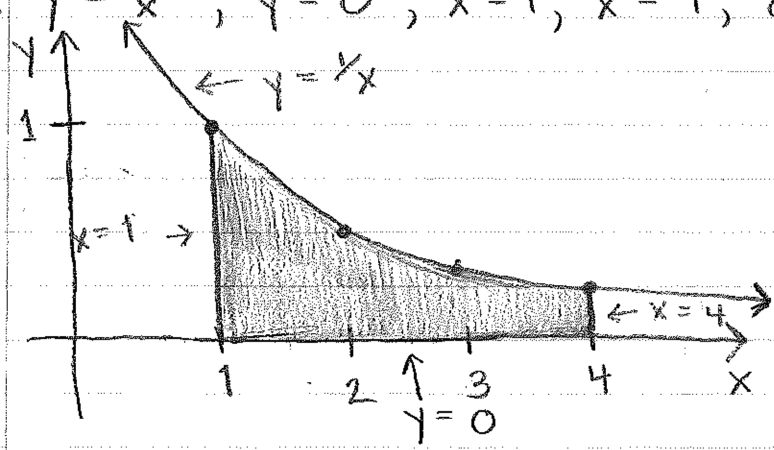
$$2\int_{0}^{1} (x-x^3)\,dx = 2\left(\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right)\Big|_0^1$$

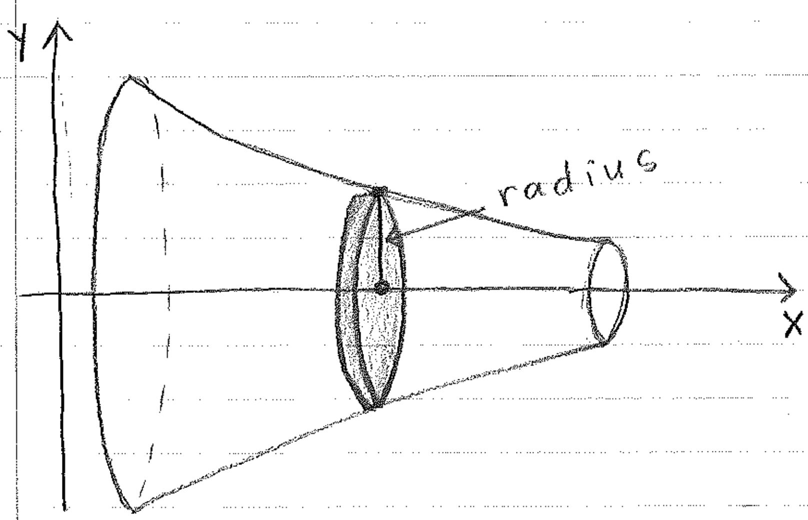
$$= 2\left(\tfrac{1}{2}(1)^2 - \tfrac{1}{4}(1)^4 - 0\right)$$

$$= 2\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = 2\left(\tfrac{1}{4}\right) = \boxed{\tfrac{1}{2}}$$

$\boxed{6.2}$

2. $y = \frac{1}{x}$, $y = 0$, $x = 1$, $x = 4$; about x-axis





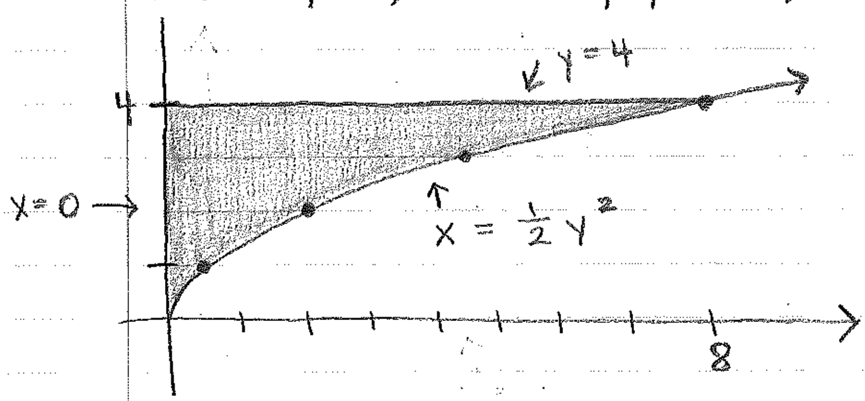
A cross-section is a disk with radius $\frac{1}{x}$, so its area is $A(x) = \pi\left(\frac{1}{x}\right)^2 = \pi x^{-2}$.

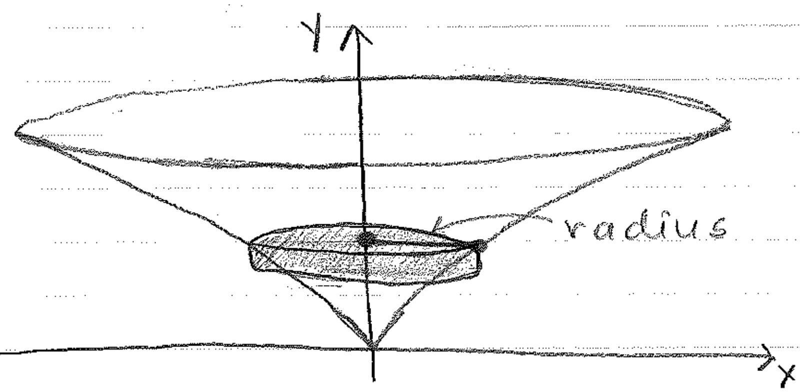
$$V = \int_1^4 A(x)\,dx = \int_1^4 \pi x^{-2}\,dx$$

$$= \pi\left[-x^{-1}\right]_1^4 = \pi\left(-\frac{1}{4} - \left(-\frac{1}{1}\right)\right)$$

$$= \pi\left(-\frac{1}{4} + 1\right) = \boxed{\frac{3}{4}\pi}$$

6. $2x = y^2$, $x = 0$, $y = 4$; about y-axis





radius $= \frac{1}{2}y^2 \implies A(y) = \pi\left[\left(\frac{1}{2}\right)y^2\right]^2$

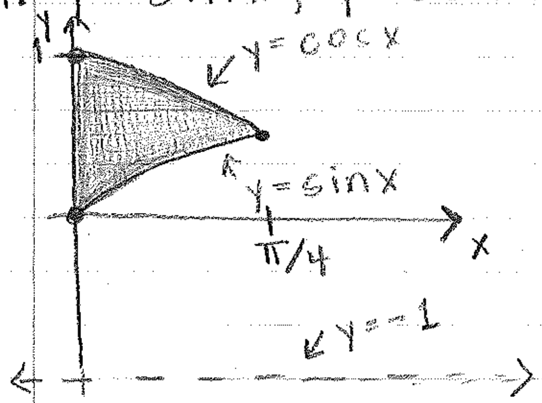
$A(y) = \frac{\pi}{4}y^4$

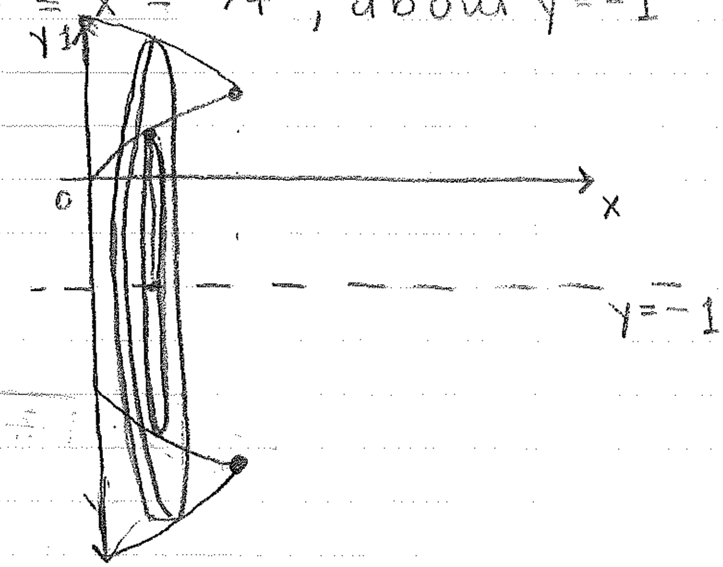
$V = \int_0^4 A(y)\,dy = \int_0^4 \frac{\pi}{4}y^4\,dy$

$= \frac{\pi}{4}\cdot\frac{1}{5}y^5\Big|_0^4 = \frac{\pi}{20}y^5\Big|_0^4$

$= \frac{\pi}{20}(4)^5 - 0 = \boxed{\frac{256\pi}{5}}$

14. $y = \sin x$, $y = \cos x$, $0 \leq x \leq \pi/4$; about $y = -1$





A cross-section is a washer with inner radius (r_1) $\sin x - (-1)$ and outer radius (r_2) $\cos x - (-1)$

$r_1 = \sin x + 1$, $r_2 = \cos x + 1$

$\implies A(x) = \pi r_2^2 - \pi r_1^2 = \pi(r_2^2 - r_1^2)$

$= \pi\left[(\cos x + 1)^2 - (\sin x + 1)^2\right]$

$= \pi(\cos^2 x + 2\cos x + \cancel{1} - \sin^2 x - 2\sin x - \cancel{1})$

<u>Trig identity</u>

$\cos^2 x - \sin^2 x = \cos 2x$

$= \pi(\cos 2x + 2\cos x - 2\sin x)$

$V = \int_0^{\pi/4} A(x)\,dx = \int_0^{\pi/4} \pi(\cos 2x + 2\cos x - 2\sin x)\,dx$

u-substitution, $u = 2x$

$$= \pi \left[\tfrac{1}{2} \sin 2x + 2 \sin x + 2 \cos x \right]_0^{\pi/4}$$

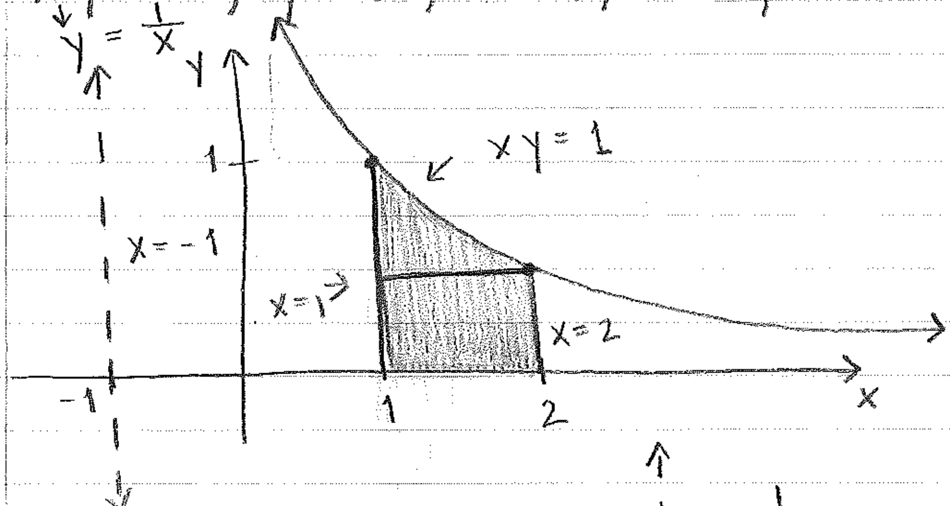
$$= \pi \left[\left(\tfrac{1}{2} \sin(2(\pi/4)) + 2 \sin(\pi/4) + 2 \cos(\pi/4) \right) - \left(\tfrac{1}{2} \sin(2(0)) + 2 \sin(0) + 2 \cos(0) \right) \right]$$

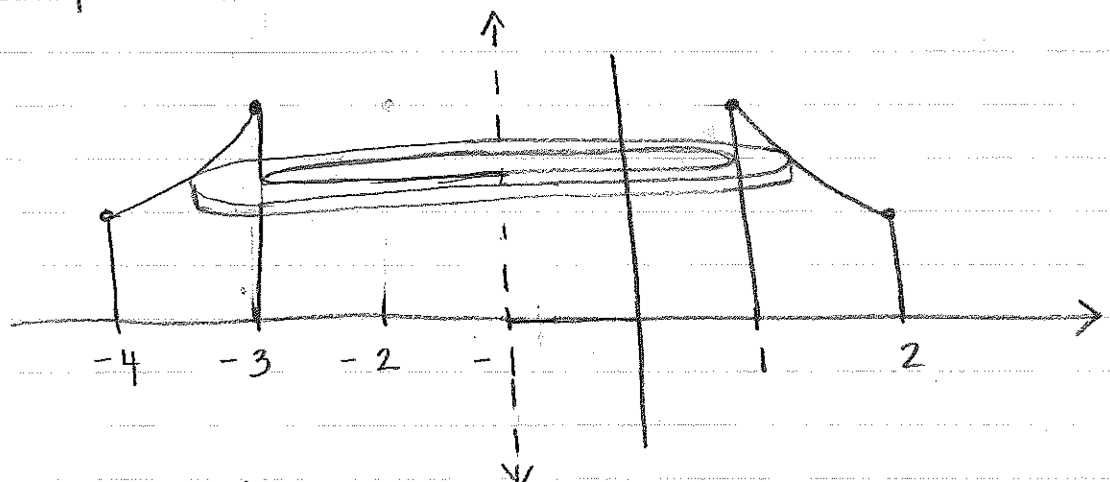
$$= \pi \left(\tfrac{1}{2} + \sqrt{2} + \sqrt{2} - 2 \right)$$

$$= \boxed{\left(2\sqrt{2} - 3/2 \right) \pi}$$

16. $xy = 1$, $y = 0$, $x = 1$, $x = 2$; about $x = -1$

$y = \frac{1}{x}$





For $0 \le y < \frac{1}{2}$, a cross-section is a washer with inner radius $1 - (-1) = 2$, and outer radius $2 - (-1) = 3$. So its area is $A(y) = \pi(3^2 - 2^2) = 5\pi$.

For $\frac{1}{2} \le y \le 1$, a cross-section is a washer with inner radius $1 - (-1) = 2$, and outer radius $\frac{1}{y} - (-1) = \frac{1}{y} + 1$.

$$\Rightarrow A(y) = \pi \left[\left(\tfrac{1}{y} + 1 \right)^2 - 2^2 \right]$$

$$= \pi \left(\tfrac{1}{y^2} + \tfrac{2}{y} + 1 - 4 \right) = \pi \left(\tfrac{1}{y^2} + \tfrac{2}{y} - 3 \right)$$

$$V = \int_0^{1/2} 5\pi\, dy + \int_{1/2}^{1} \pi\left(\frac{1}{y^2} + \frac{2}{y} - 3\right) dy$$

$$= 5\pi y \Big|_0^{1/2} + \pi\left[-\frac{1}{y} + 2\ln|y| - 3y\right]_{1/2}^{1}$$

$$= 5\pi\left(\tfrac{1}{2}\right) + \pi\left[-1 + 2\ln(1) - 3 - \left(-2 + 2\ln\left(\tfrac{1}{2}\right) - \tfrac{3}{2}\right)\right]$$

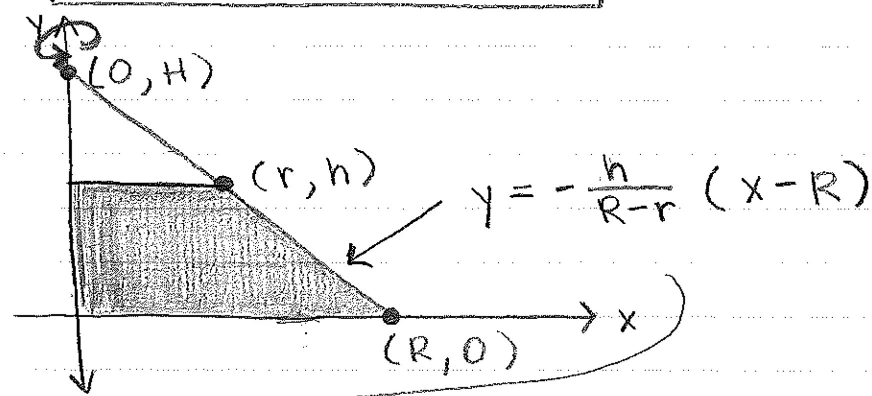
$$= \frac{5\pi}{2} + \pi\left(-4 + 2 - 2\ln\left(\tfrac{1}{2}\right) + \tfrac{3}{2}\right)$$

$$= \tfrac{5}{2}\pi + \pi\left(-\tfrac{1}{2} - 2\ln\left(\tfrac{1}{2}\right)\right)$$

$$= 2\pi + 2\pi\ln(2)$$

$$= \boxed{2\pi(1 + \ln(2))}$$

48.



$$-\frac{R-r}{h} y = x - R$$

$$\Rightarrow x = R - \frac{R-r}{h} y$$

$$\text{radius} = R - \frac{R-r}{h} y.$$

$\Rightarrow$

$$V = \pi \int_0^h \left(R - \frac{R-r}{h} y\right)^2 dy$$

$$= \pi \int_0^h \left(R^2 - \frac{2R(R-r)}{h} y + \left(\frac{R-r}{h}\right)^2 y^2\right) dy$$

$$= \pi\left[R^2 y - \frac{R(R-r)}{h} y^2 + \frac{1}{3}\left(\frac{R-r}{h}\right)^2 y^3 \right|_0^h$$

$$= \pi\left(R^2 h - R(R-r)h + \tfrac{1}{3}(R-r)^2 h\right)$$

$$= \tfrac{1}{3}\pi h\left(3R^2 - 3R^2 + 3Rr + R^2 - 2Rr + r^2\right)$$

$$= \boxed{\frac{1}{3}\pi h\left(R^2 + Rr + r^2\right)}$$