MATH 172 Homework 8

7.8 # 49, 50, 51, 58, 69, 73, 74, 80

49. $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$

For $x > 0$, $\dfrac{x}{x^3+1} < \dfrac{x}{x^3} = \dfrac{1}{x^2}$

$\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by Theorem 2 with $p = 2 > 1$.

so $\displaystyle\int_1^\infty \frac{x}{x^3+1}\,dx$ is convergent by the Comparison Theorem.

$$\int_0^\infty \frac{x}{x^3+1}\,dx = \int_0^1 \frac{x}{x^3+1}\,dx + \int_1^\infty \frac{x}{x^3+1}\,dx$$

$\displaystyle\int_0^1 \frac{x}{x^3+1}\,dx$ is a constant (finite).

Therefore, $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$ is convergent.

50. $\displaystyle\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,dx$

$0 \le \sin^2 x \le 1$

$1 \le 1+\sin^2 x \le 2$

So, $\dfrac{1+\sin^2 x}{\sqrt{x}} \ge \dfrac{1}{\sqrt{x}}$

$\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is divergent by Theorem 2 with $p = 1/2 \le 1$.

So, $\displaystyle\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,dx$ is divergent by the Comparison Theorem.

51. $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$

$f(x) = \dfrac{x+1}{\sqrt{x^4-x}} > \dfrac{x+1}{\sqrt{x^4}} > \dfrac{x}{x^2} = \dfrac{1}{x}$, for $x > 1$

$f(x)$ has a discontinuity at $x = 1$, so we will break up the integral:

$$\int_1^\infty f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^\infty f(x)\,dx.$$

$\displaystyle\int_2^\infty \frac{1}{x}\,dx$ is divergent from Theorem 2 with $p = 1 \le 1$.

Therefore, $\displaystyle\int_2^\infty f(x)\,dx$ is also divergent from the Comparison Theorem.

Therefore $\displaystyle\int_1^\infty f(x)\,dx$ diverges.

58. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^p}\,dx \rightarrow \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^p}\,dx$

Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

@ $x = e$, $u = 1$; @ $x = t$, $u = \ln t$

$\displaystyle\lim_{t\to\infty} \int_1^{\ln t} \frac{1}{u^p}\,dt = \frac{u^{1-p}}{1-p}\Big|_1^{\ln t}$

$\displaystyle\lim_{t\to\infty} \frac{1}{1-p}\left((\ln t)^{1-p} - 1\right)$

$\displaystyle\lim_{t\to\infty} (\ln t)^{1-p}$ converges to 0

when $1-p < 0 \Rightarrow p > 1$.

when $p > 1$, converges to $\dfrac{1}{p-1}$.

69. $\gamma = \int_0^{\infty} \frac{cN(1-e^{-kt})}{k} e^{-\lambda t}\, dt$

pull constants out front

$\frac{cN}{k} \lim_{x\to\infty} \int_0^x (1-e^{-kt})e^{-\lambda t}\, dt$

$\frac{cN}{k} \lim_{x\to\infty} \int_0^x \left(e^{-\lambda t} - e^{(-k-\lambda)t}\right) dt$

$\frac{cN}{k} \lim_{x\to\infty} -\frac{1}{\lambda} e^{-\lambda t} + \frac{1}{k+\lambda} e^{(-k-\lambda)t} \Big|_0^x$

$\frac{cN}{k} \lim_{x\to\infty} -\frac{1}{\lambda} e^{-\lambda x} + \frac{1}{(k+\lambda)e^{(k+\lambda)x}} - \left(-\frac{1}{\lambda} + \frac{1}{k+\lambda}\right)$

$\frac{cN}{k}\left(\frac{1}{\lambda} - \frac{1}{k+\lambda}\right) = \frac{cN}{k}\left(\frac{k+\lambda-\lambda}{\lambda(k+\lambda)}\right)$

$= \boxed{\frac{cN}{\lambda(k+\lambda)}}$

73. $F(s) = \int_0^{\infty} f(t) e^{-st}\, dt$

a) $F(s) = \int_0^{\infty} e^{-st}\, dt$

$\lim_{n\to\infty} \int_0^n e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_0^n$

$\lim_{n\to\infty} -\frac{1}{s}\left(e^{-sn} - e^{0}\right) = \frac{1}{s}$ if $s > 0$

$\boxed{F(s) = \frac{1}{s} \text{ with domain } \{s \mid s > 0\}}$

b) $F(s) = \int_0^{\infty} e^t e^{-st}\,dt \rightarrow \lim_{x\to\infty} \int_0^x e^{(1-s)t}\,dt$

$\lim_{x\to\infty} \frac{1}{1-s} e^{(1-s)t} \Big|_0^x$

$\lim_{x\to\infty} \frac{1}{1-s}\left(e^{(1-s)x} - e^{(1-s)0}\right) = \frac{1}{s-1}$, $1-s<0$
$(s>1)$

$\boxed{F(s) = \frac{1}{s-1} \text{ with domain } \{s \mid s>1\}}$

c) $F(s) = \int_0^{\infty} t e^{-st}\,dt \rightarrow \lim_{x\to\infty} \int_0^x t e^{-st}\,dt$

Integration by parts:

$u = t \qquad dv = e^{-st}\,dt$

$du = dt \qquad v = -\frac{1}{s} e^{-st}$

$\lim_{x\to\infty} -\frac{1}{s} t e^{-st} \Big|_0^x + \frac{1}{s}\int_0^x e^{-st}\,dt$

$\lim_{x\to\infty} -\frac{1}{s} x e^{-sx} + 0 + \frac{1}{s}\left(-\frac{1}{s} e^{-st}\right)\Big|_0^x$

$\lim_{x\to\infty} -\frac{1}{s}\cdot\frac{x}{e^{sx}} + \lim_{x\to\infty} -\frac{1}{s^2}\left(e^{-sx} - e^{s(0)}\right)$

L'Hopital's Rule

$-\frac{1}{s} \lim_{x\to\infty} \frac{1}{se^{sx}} + \lim_{x\to\infty} -\frac{1}{s^2}\left(\frac{1}{e^{sx}} - 1\right)$

$= \frac{1}{s^2}$ if $s > 0$.

$\boxed{F(s) = \frac{1}{s^2} \text{ with domain } \{s \mid s>0\}}$

74. $0 \le f(t) \le Me^{at}$

Multiply by e^{-st}

$0 \le f(t)e^{-st} \le Me^{at}e^{-st}$

Now use the Comparison Theorem

$$\int_0^\infty Me^{at}e^{-st} \rightarrow \lim_{x\to\infty} M\int_0^x e^{(a-s)t}\,dt$$

$$M \lim_{x\to\infty} \frac{1}{a-s} e^{(a-s)t}\Big|_0^x$$

$$M \lim_{x\to\infty} \frac{1}{a-s}\left(e^{(a-s)x} - e^{(a-s)0}\right)$$

$$= \frac{M}{s-a} \quad \text{if} \quad a-s<0 \Rightarrow s>a.$$

Therefore, by the Comparison Theorem, $F(s) = \int_0^\infty f(t)e^{-st}\,dt$

is also convergent for $s > a$.

80. $$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{c}{3x+1}\right)dx \leftarrow ★$$

$$\rightarrow \lim_{t\to\infty} \int_0^t \frac{x}{x^2+1}\,dx - c\int_0^t \frac{1}{3x+1}\,dx$$

Let $u = x^2+1$
$du = 2x\,dx$
$\Rightarrow x\,dx = \frac{1}{2}du$
@ $x=0$, $u=1$
@ $x=t$, $u=t^2+1$

Let $u = 3x+1$
$du = 3\,dx$
$\Rightarrow dx = \frac{1}{3}du$
@ $x=0$, $u=1$
@ $x=t$, $u=3t+1$

$$\lim_{t\to\infty} \frac{1}{2}\int_1^{t^2+1} \frac{1}{u}\,du - \frac{c}{3}\int_1^{3t+1}\frac{1}{u}\,du$$

$$\lim_{t\to\infty} \frac{1}{2}\ln|u|\Big|_1^{t^2+1} - \frac{c}{3}\ln|u|\Big|_1^{3t+1}$$

$$\lim_{t\to\infty} \frac{1}{2}\ln(t^2+1) - \cancel{\ln(1)}^{\,0} - \frac{c}{3}\left(\ln|3t+1| - \cancel{\ln(1)}^{\,0}\right)$$

$$\lim_{t\to\infty} \ln\left(\frac{(t^2+1)^{1/2}}{(3t+1)^{c/3}}\right) = \ln\left(\lim_{t\to\infty} \frac{\sqrt{t^2+1}}{(3t+1)^{c/3}}\right)$$

$$L = \lim_{t\to\infty} \frac{\sqrt{t^2+1}}{(3t+1)^{c/3}}$$, L'Hopital's Rule

$$= \lim_{t\to\infty} \frac{t/\sqrt{t^2+1}}{c\,(3t+1)^{c/3-1}} \leftarrow \circledast$$

$$\circledast = \lim_{t\to\infty} \frac{t}{\sqrt{t^2+1}} = 1$$

$$L = \frac{1}{c} \lim_{t\to\infty} \frac{1}{(3t+1)^{c/3-1}}$$

$= 0$ if $c/3 - 1 > 0 \Rightarrow c > 3$

$= 1$ if $c/3 - 1 = 0 \Rightarrow c = 3$

$= \infty$ if $c/3 - 1 < 0 \Rightarrow c < 3$

$\ln(0) = -\infty$
$\ln(1) = 0$
$\ln(\infty) = \infty$

$\Rightarrow$ ★ converges only if $c = 3$.