MATH 172, FALL 2017
EXAM I-VERSION A/B

LAST NAME(print): ______________________ FIRST NAME(print): ______________________

SECTION NUMBER: ____________

DIRECTIONS:

1. The use of a calculator, laptop or computer is prohibited.
2. TURN OFF cell phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.
3. This exam consists of 11 multiple choices questions worth 4 points each for a total of 44 points and 4 work out problems worth 14 points each for a total of 56 points.
4. In Part A (Problems 1-11), mark the correct choice on your ScanTron using a No. 2 pencil. *For your own records, also record your choices on your exam!*
5. In Part B (Problems 12-15), present your solutions in the space provided. *Show all your work* neatly and concisely and *clearly indicate your final answer by boxing it.* You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.
6. You must turn in both your exam and your scantron.
7. Be sure to *write your name, section number and version letter of the exam on the ScanTron form.*

THE AGGIE CODE OF HONOR

"An Aggie does not lie, cheat or steal, or tolerate those who do."

Signature: KEY.

Problems	Points Awarded	Points
1-11		44
12		14
13		14
14		14
15		14
TOTAL		100

MC:

1. C
2. A
3. B
4. B
5. D
6. C
7. D
8. A
9. B
10. C
11. C

13. (14 points)

(a) (6 points) Give all the antiderivatives of the function $f(y) = y\ln(y)$.

(Integration by parts)

$$\int y\ln(y)\,dy = \frac{y^2}{2}\ln(y) - \int \frac{y}{2}\,dy = \frac{y^2}{2}\ln(y) - \frac{y^2}{4} + C$$

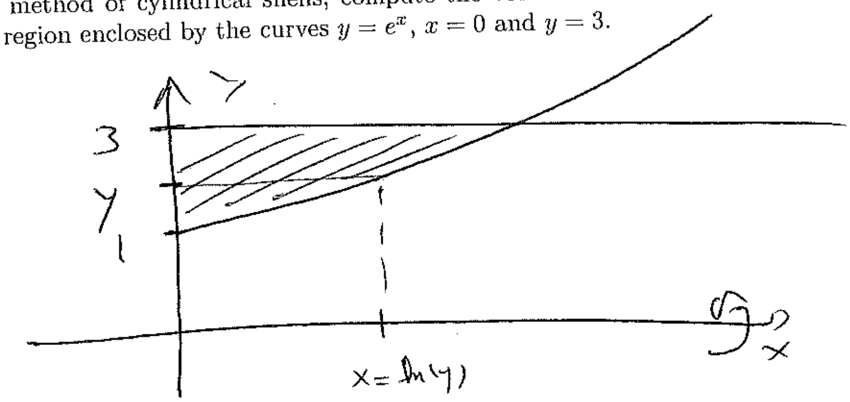
$u(y) = \ln(y) \qquad v'(y) = y$

$u'(y) = \frac{1}{y} \qquad v(y) = \frac{y^2}{2}$

$$\boxed{\int y\ln(y)\,dy = \frac{y^2}{2}\ln(y) - \frac{y^2}{4} + C}$$

(b) (8 points) Using the method of cylindrical shells, compute the volume V of the solid obtained by rotating about the x-axis the region enclosed by the curves $y = e^x$, $x = 0$ and $y = 3$.

$e^x = y \Leftrightarrow x = \ln y$



$$V = \int_1^3 2\pi y\ln(y)\,dy = 2\pi\int_1^3 y\ln(y)\,dy$$

$$= 2\pi\left[\frac{y^2}{2}\ln(y) - \frac{y^2}{4}\right]_1^3 = 2\pi\left[\frac{9}{2}\ln 3 - \frac{9}{4} - \left(0 - \frac{1}{4}\right)\right]$$

$$= 2\pi\left(\frac{9}{2}\ln(3) - \frac{8}{4}\right) = \pi(9\ln 3 - 4)$$

$$\boxed{V = \pi(9\ln 3 - 4)}$$

PART B

12. (14 points)

(a) (4 points) State the integration by parts theorem.

Let $f, g : [a,b] \to \mathbb{R}$ such that

1) f, g are differentiable on $[a,b]$,

2) f', g' are integrable on $[a,b]$,

then $\int_a^b f'(x)g(x)dx = \left[f(x)g(x)\right]_a^b - \int_a^b f(x)g'(x)dx$

(b) (10 points) Let f and g be twice differentiable functions on $[0,2]$ such that f'' and g'' are continuous on $[0,2]$. If $f(0) = g(0) = 0$, show that $\int_0^2 f(x)g''(x)dx = f(2)g'(2) - f'(2)g(2) + \int_0^2 f''(x)g(x)dx$

Since f and g are twice differentiable then f' and g' are integrable on $[0,2]$ and we can apply the IBP theorem.

$$\int_0^2 f(x)g''(x)dx = \left[f(x)g'(x)\right]_0^2 - \int_0^2 f'(x)g'(x)dx$$

$$= f(2)g'(2) - f(0)g(0) - \int_0^2 f'(x)g'(x)dx$$

$$= f(2)g'(2) - \int_0^2 f'(x)g'(x)dx \quad \text{since } f(0)=0.$$

Since f'' and g'' are continuous on $[0,2]$ they are integrable on $[0,2]$ and we can apply the IBP theorem again.

$$\int_0^2 f(x)g''(x)dx = f(2)g'(2) - \left(\left[f'(x)g(x)\right]_0^2 - \int_0^2 f''(x)g(x)dx\right)$$

$$= f(2)g'(2) - \left(f'(2)g(2) - f'(0)g(0) - \int_0^2 f''(x)g(x)dx\right)$$

$$= f(2)g'(2) - f'(2)g(2) - \int_0^2 f''(x)g(x)dx \quad \text{since } g(0)=0.$$

14. (14 points)

(a) (4 points) If $x = \dfrac{3}{2\cos(\theta)}$ with $0 \leq \theta < \frac{\pi}{2}$. Express $\sin(\theta)$ in terms of x.

$\cos(\theta) = \dfrac{3}{2x}$

$\sin(\theta) = \dfrac{\sqrt{4x^2-9}}{2x} = \sqrt{1 - \dfrac{9}{4x^2}}$

$$\boxed{\sin(\theta) = \sqrt{1 - \frac{9}{4x^2}}}$$

(b) (10 points) Give all the antiderivatives of the function $f(x) = \dfrac{1}{x^2\sqrt{4x^2-9}}$

$$I = \int \frac{dx}{x^2\sqrt{4x^2-9}} = \int \frac{dx}{x^2\sqrt{4\left(x^2 - \left(\frac{3}{2}\right)^2\right)}} = \frac{1}{2}\int \frac{dx}{x^2\sqrt{x^2-\left(\frac{3}{2}\right)^2}}$$

$$x = \frac{3}{2\cos(\theta)} \qquad \frac{dx}{d\theta} = \frac{3}{2}\,\frac{\sin(\theta)}{\cos^2(\theta)}$$

$$I = \frac{1}{2}\int \frac{\frac{3}{2}\frac{\sin\theta}{\cos^2(\theta)}\,d\theta}{\frac{9}{4\cos^2(\theta)}\sqrt{\left(\frac{3}{2}\right)^2\left(\frac{1}{\cos^2\theta}-1\right)}}$$

$$= \frac{3}{4}\,\frac{4}{9}\int \frac{\sin\theta}{\cos^2(\theta)}\,\frac{\cos^2(\theta)\,d\theta}{\frac{3}{2}\sqrt{\tan^2(\theta)}}$$

$$= \frac{2}{9}\int \frac{\sin\theta}{\tan\theta}\,d\theta = \frac{2}{9}\int \frac{\sin\theta}{\frac{\sin\theta}{\cos(\theta)}}\,d\theta$$

$$= \frac{2}{9}\int \cos(\theta)\,d\theta = \frac{2}{9}\sin(\theta) + C$$

$$= \frac{2}{9}\sqrt{1-\frac{9}{4x^2}} + C, \qquad \boxed{\int \frac{dx}{x^2\sqrt{4x^2-9}} = \frac{2}{9}\sqrt{1-\frac{9}{4x^2}} + C}$$

15. (14 points)

(a) (4 points) Let $t = \tan(\frac{\theta}{2})$ with $-\pi < \theta < \pi$. Compute $\dfrac{dt}{d\theta}$ in terms of t.

$$\frac{dt}{d\theta} = \left(1 + \tan^2\left(\frac{\theta}{2}\right)\right)\frac{1}{2} \quad \text{(chain rule)}$$

$$= \frac{1+t^2}{2}$$

$$\boxed{\frac{dt}{d\theta} = \frac{1+t^2}{2}}$$

(b) (4 points) Let $t = \tan(\frac{\theta}{2})$ with $-\pi < \theta < \pi$. Show that $\cos(\theta) = \dfrac{1-t^2}{1+t^2}$.
(Hint: You can use that $\cos(\theta) = 2\cos^2(\frac{\theta}{2}) - 1$.)

$$t^2 = \frac{\sin^2(\frac{\theta}{2})}{\cos^2(\frac{\theta}{2})} = \frac{1-\cos^2(\frac{\theta}{2})}{\cos^2(\frac{\theta}{2})} = \frac{1}{\cos^2(\frac{\theta}{2})} - 1$$

$$\Rightarrow \cos^2\left(\frac{\theta}{2}\right) = \frac{1}{1+t^2} \quad \text{and} \quad \cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$$

(c) (6 points) Using Weierstrass substitution give all the antiderivatives of the function $f(\theta) = \dfrac{1}{\cos(\theta)}$.

$$\int \frac{d\theta}{\cos(\theta)} \underset{\uparrow}{=} \int \frac{1+t^2}{1-t^2}\,\frac{2}{1+t^2}\,dt = \int \frac{2\,dt}{(1-t)(1+t)}$$

$t = \tan(\frac{\theta}{2})$ (Weierstrass substitution)

But $\dfrac{2}{(1-t)(1+t)} = \dfrac{A}{1-t} + \dfrac{B}{1+t}$ (partial fraction expansion)

and thus, $\begin{cases} A+B = 2 \\ A-B = 0 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 1 \end{cases}$

$$\int \frac{2\,dt}{(1-t)(1+t)} = \int \frac{dt}{1+t} + \int \frac{dt}{1-t} = \ln|1+t| - \ln|1-t| + C = \ln\left|\frac{1+t}{1-t}\right| + C$$

$$\boxed{\int \frac{d\theta}{\cos(\theta)} = \ln\left|\frac{1+\tan(\frac{\theta}{2})}{1-\tan(\frac{\theta}{2})}\right| + C}$$