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Invited Comment

The gravitational field of an infinite flat slab

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Abstract
We study Einstein's equations with a localized plane-symmetric source, with close attention to gauge freedom/fixing and to listing all physically distinct solutions. In the vacuum regions there are only two qualitatively different solutions, one curved and one flat; in addition, on each of the two sides there is a free parameter describing how the slab is embedded into the vacuum region. Surprisingly, for a generic slab source the solution must be curved on one side and flat on the other. We treat infinitely thin slabs in full detail and indicate how thick slabs can increase the variety of external geometry pairs. Positive energy density seems to force external geometries with curvature singularities at some distance from the slab; we speculate that such singularities occur in regions where the solution cannot be physically relevant anyway.

Keywords: Einstein equation, plane symmetry, gauge fixing

1. Introduction

General coordinate invariance is a powerful, yet tricky, feature of general relativity. Nowhere is this clearer than in solving the gravitational field equations under assumed conditions of symmetry. Consider, for instance, the gravitational field of a static, spherically symmetric star [1, chapter 10]. One's first appeal to general covariance is to choose, without loss of generality, a spherical coordinate system centered on the star; the line element must have the form

$$\mathrm{d}s^2 = -\mathrm{e}^{2\Phi}\,\mathrm{d}t^2 + \mathrm{e}^{2\Lambda}\,\mathrm{d}r^2 + \mathrm{e}^{2\Psi}\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \qquad (1)$$

with Φ, Λ, Ψ functions of r only. But this insight does not completely use up the freedom to choose coordinates. Our interest here is in redefinitions of the radial coordinate that change the symbolic form of (1) while, of course, leaving unchanged the geometry it describes. The freedom to introduce $\rho = f(r)$, where f is a differentiable monotonic function, enables one to impose any one of various relations upon the three coefficient functions. In particular:

- One could require that $\mathrm{e}^{2\Lambda}\,\mathrm{d}r^2 = \mathrm{d}\rho^2$. This means that differences in ρ directly give the physical proper distances between points on a radial curve.
- One can require that $\mathrm{e}^{2\Psi} = \rho^2$. This, the most common choice, means that $2\pi\rho$ is the circumference of the sphere at constant ρ, and more generally that physical proper distances in angular directions are accurately represented by the Euclidean formulas.
- One could require that $\mathrm{e}^{2\Psi} = \rho^2\mathrm{e}^{2\Lambda}(\mathrm{d}r/\mathrm{d}\rho)^2$. This means that the spatial part of the metric is

$$\rho^{-2}e^{2\Psi}\left[\mathrm{d}\rho^2 + \rho^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] = \rho^{-2}\mathrm{e}^{2\Psi}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right), \qquad (2)$$

(where Ψ is now a new function of ρ). This isotropic system is often used to represent the local physics near a point with minimal distortion.

Each of these conventions is useful in particular circumstances, so there is no uniquely optimal coordinate system. However, to carry out a correct, detailed calculation, one much pick a convention and use it consistently. The point of the example (1) is that there are only two independent degrees of freedom in the problem, not the three that superficially appear in the general form of the solution. Neglecting this 'gauge freedom' can lead to errors and confusion. Solutions that look different may be physically the same. Apparent phenomena, such as singularities or new modes of oscillation, may be gauge artifacts. Mixing requirements associated with different gauges may cause a legitimate solution to be incorrectly excluded, and adhering too rigidly to a favorite gauge condition may cause a solution to be overlooked.

Richard Arnowitt, Stanley Deser, and Charles Misner [2, chapter 7] developed the most successful analysis of gauge invariance and gauge fixing in general gravitodynamics. The spherical star problem and the similar problem treated in the present paper belong to the simpler domain of gravitostatics, specialized moreover to particular symmetries, but the theme is the same. One must isolate the true degrees of freedom from coordinate artifacts, without losing the power that comes from the option of switching to a different coordinate system when it is better adapted to the problem at hand.

A previous paper [3], and previous literature going back to Weyl and Levi–Civita around 1918, considered the analog of the relativistic star problem under the assumption of axial and cylindrical symmetry. Here we consider the seemingly more elementary problem where the static matter source lies in or near a plane, with rotational and translational symmetry in that plane. Not surprisingly, this problem also has a long history (see [4]); it, also, traces back to Levi–Civita in 1918. To the best of our knowledge, however, there is no previous treatment of this configuration that clearly and systematically treats the problem of gauge fixing and also constructs the most general global solutions possessing the symmetries of the source.

The original impetus to study this problem came from the theory of vacuum energy. Idealized boundary conditions in Casimir-type problems give rise to locally finite energy densities which integrate to infinite energies on the walls; although it is tempting to say 'renormalization' and ignore these divergent terms, it is more widely believed that in a more realistic theory they will be finite but nonzero and will have physical meaning [5]. In particular, in principle the vacuum energy density near the boundary should be included in the source of the Einstein equation. It is implausible that the vacuum energy in any real experiment could significantly influence the gravitational field, but modifying the divergent theory in a physically consistent and plausible way has turned out to be nontrivial [6]. As a matter of principle, it is of interest to study the statics of the gravitational field, the quantized matter field, and the classical source (the matter constituting the boundary) in the fully coupled theory. Here we leave out the quantum field and study the classical problem of the gravitational field of a planar boundary alone.

In sections 2–4 we set up the problem and solve the equations for vacuum. The results in these sections are not new, but our procedure and notation have advantages. In section 5 we construct the solutions for a delta-function source (energy density and pressure) on a plane. Unlike previous authors, we allow reflection-asymmetric solutions and find that they are necessary to accommodate all physically reasonable sources. In section 6 we briefly investigate generalizations to sources of finite thickness.

2. The scenario

We assume either an idealized matter source confined to a plane $z = 0$ or a nonsingular source confined to a layer $|z| < z_0$. The matter distribution is invariant under translations, rotations, and reflections in the $x - y$ plane and under time translations and time reversal. We consider only solutions of the Einstein field equation that share these symmetries (i.e., do not contain gravitational waves, for instance). We do not assume that the solution is invariant under the space reflection $z \to -z$, even when the source is. (The assumed symmetries in the other three coordinates are sufficient to exclude off-diagonal metric components.) We do not assume that the scenario is invariant under Lorentz transformations parallel to the slab.

It follows that the line element has the form

$$\begin{aligned} \mathrm{d}s^2 = -\mathrm{e}^{2\Phi(z)}\mathrm{d}t^2 + \mathrm{e}^{2\Psi(z)}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) \\ + \mathrm{e}^{2\Lambda(z)}\mathrm{d}z^2, \end{aligned} \tag{3}$$

where the three coefficient functions depend only on z. Similarly, the stress tensor has three independent components, which are functions only of z , and

$$T^\mu{}_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p_z \end{pmatrix} \tag{4}$$

displays the energy density and the two pressures in a local orthonormal frame. (We use the now most standard sign conventions [1, 7], in which g_{00} is negative and T_{00} is positive, and the Riemann and Ricci tensors are given by definitions quoted below.)

In analogy with the spherical scenario discussed previously, the precise definition of the z coordinate is an important decision of gauge fixing, and the most natural choices are those that simplify (3) so that it involves two independent functions in a natural way. The most obvious procedure is to construct the proper distance z' by solving

$$\mathrm{d}z' = \mathrm{e}^{\Lambda}\,\mathrm{d}z \tag{5}$$

and then to discard the primes—in other words, to assume from the start that $\Lambda = 0$. However, other gauge conditions, such as $\Lambda = \Psi$ (isotropic) or $\Lambda = \Phi$ (conformally flat in the t-z plane), may be useful in certain circumstances.

3. Curvature tensor and Einstein equation

We compute all the standard ingredients from the metric (3), keeping the gauge general for the moment.

3.1. Christoffel symbols

$$\Gamma^\alpha{}_{\beta\delta} = \frac{1}{2}g^{\alpha\gamma}\left(g_{\beta\gamma,\delta} + g_{\delta\gamma,\beta} - g_{\beta\delta,\gamma}\right)$$

$$\Gamma^0{}_{03} = \Phi', \tag{6}$$

$$\Gamma^1{}_{13} = \Gamma^2{}_{23} = \Psi', \tag{7}$$

$$\Gamma^3{}_{33} = \Lambda', \tag{8}$$

$$\Gamma^3{}_{00} = \Phi' \mathrm{e}^{2(\Phi-\Lambda)}, \tag{9}$$

$$\Gamma^3{}_{11} = \Gamma^3{}_{22} = -\Psi' \mathrm{e}^{2(\Psi-\Lambda)}. \tag{10}$$

3.2. Riemann tensor

$$R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha{}_{\beta\delta} - \partial_\delta \Gamma^\alpha{}_{\beta\gamma} + \Gamma^\mu{}_{\beta\delta}\Gamma^\alpha{}_{\mu\gamma} - \Gamma^\mu{}_{\beta\gamma}\Gamma^\alpha{}_{\mu\delta}$$

$$R^1{}_{010} = R^2{}_{020} = \Phi'\Psi' \mathrm{e}^{2(\Phi-\Lambda)}, \tag{11}$$

$$R^3{}_{030} = \left[\Phi'' + \Phi'^2 - \Phi'\Lambda'\right] \mathrm{e}^{2(\Phi-\Lambda)}, \tag{12}$$

$$R^0{}_{101} = R^0{}_{202} = -\Phi'\Psi' \mathrm{e}^{2(\Psi-\Lambda)}, \tag{13}$$

$$R^2{}_{121} = R^1{}_{212} = -(\Psi')^2 \mathrm{e}^{2(\Psi-\Lambda)}, \tag{14}$$

$$R^3{}_{131} = R^3{}_{232} = -\left[\Psi'' + \Psi'^2 - \Psi'\Lambda'\right] \mathrm{e}^{2(\Psi-\Lambda)}, \tag{15}$$

$$R^0{}_{303} = -\Phi'' + \Lambda'\Phi' - \Phi'^2, \tag{16}$$

$$R^1{}_{313} = R^2{}_{323} = -\Psi'' + \Lambda'\Psi' - \Psi'^2. \tag{17}$$

Components in which an index appears only once are 0 by virtue of the reflection symmetries.

3.3. Ricci tensor

$$R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}$$

$$R_{00} = \left[\Phi'' + \Phi'^2 - \Phi'\Lambda' + 2\Phi'\Psi'\right] \mathrm{e}^{2(\Phi-\Lambda)}, \tag{18}$$

$$R_{11} = R_{22} = -\left[\Psi'' + 2\Psi'^2 - \Psi'\Lambda' + \Phi'\Psi'\right] \mathrm{e}^{2(\Psi-\Lambda)}, \tag{19}$$

$$R_{33} = -\Phi'' + \Lambda'\Phi' - \Phi'^2 + 2\left(-\Psi'' + \Lambda'\Psi' - \Psi'^2\right). \tag{20}$$

Off-diagonal components vanish identically.

3.4. Einstein field equations

Rather than tabulate the formulas for the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R^\alpha{}_\alpha g_{\mu\nu}$, we recall that the Einstein equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ (in natural units) is equivalent in space-time dimension 4 to

$$R_{\mu\nu} = 8\pi \tilde{T}_{\mu\nu}, \tag{21}$$

$$\tilde{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2} T^\alpha{}_\alpha g_{\mu\nu}. \tag{22}$$

From (20), (4), (3) we then have

$$\left(\Phi'' + \Phi'^2 - \Phi'\Lambda' + 2\Phi'\Psi'\right) \mathrm{e}^{-2\Lambda} = 4\pi\left(\rho + 2p + p_z\right), \tag{23}$$

$$-\left(\Psi'' + 2\Psi'^2 - \Psi'\Lambda' + \Phi'\Psi'\right) \mathrm{e}^{-2\Lambda} = 4\pi(\rho - p_z), \tag{24}$$

$$\left(-\Phi'' + \Lambda'\Phi' - \Phi'^2 - 2\Psi'' + 2\Lambda'\Psi' - 2\Psi'^2\right) \mathrm{e}^{-2\Lambda} = 4\pi\left(\rho - 2p + p_z\right). \tag{25}$$

3.5. Conservation law and constraint equation

The stress tensor must satisfy

$$T^{\alpha\beta}{}_{;\beta} \equiv \partial_\beta T^{\alpha\beta} + \Gamma^\alpha{}_{\mu\beta} T^{\mu\beta} + \Gamma^\beta{}_{\mu\beta} T^{\alpha\mu} = 0.$$

The only nontrivial component is the one with $\alpha = 3$:

$$\partial_z p_z \equiv p_z{}' = -\left(\rho + p_z\right)\Phi' + 2(p - p_z)\Psi'. \tag{26}$$

The conservation law (26) constrains the behavior of the source, rather than the metric. Nevertheless, it is a consequence of the field equations (23)–(25) and the Bianchi identity and may be used to replace the most complicated of the field equations. (Compare the similar analyses of the spherical [1, chapter 10], cylindrical [3], and cosmological [1, chapter 12] scenarios.) This situation reflects the fact that our three Einstein equations do not contain Λ'' and hence are not really independent dynamical equations. An equation with no second derivatives at all can be obtained by adding (23) and (25) and subtracting twice (24):

$$8\pi p_z = \left(\Psi'^2 + 2\Phi'\Psi'\right) \mathrm{e}^{-2\Lambda}. \tag{27}$$

Now (25) is a linear combination of (23), (24), and (27), so it henceforth can be ignored as redundant. The equation system reduces to second-order equations (23) and (24) for Φ and Ψ and (27) as a constraint on their initial data. (The derivative of (27) agrees with (26) when Φ'', Ψ'', and p_z are eliminated via the three equations of the system, proving consistency.)

The field equations give no information to determine Λ. This confirms the expectation that Λ is 'pure gauge' and must be fixed by some rather arbitrary extra condition.

3.6. Decoupling

Add (24) to half of (27):

$$-\Psi'' - \frac{3}{2}\Psi'^2 = 4\pi\rho \mathrm{e}^{+2\Lambda}. \tag{28}$$

This equation is independent of Φ and is a first-order ordinary differential equation for Ψ', given ρ and Λ. In the vacuum case it is separable, hence solvable. Then, insofar as Ψ is known, (23) becomes a similar equation for Φ', given p. After elimination of p_z via (27), it becomes

$$\Phi'' + \Phi'^2 - \Lambda'\Phi' + \Psi'\Phi' - \frac{1}{2}\Psi'^2 = 4\pi(\rho + 2p)\mathrm{e}^{+2\Lambda}. \tag{29}$$

Hence we can take the field equations of the problem to be (27), (28) and (29) (or (26) with initial data satisfying (27)), with the input data Λ, ρ, and p along with suitable initial data for the three unknown functions.

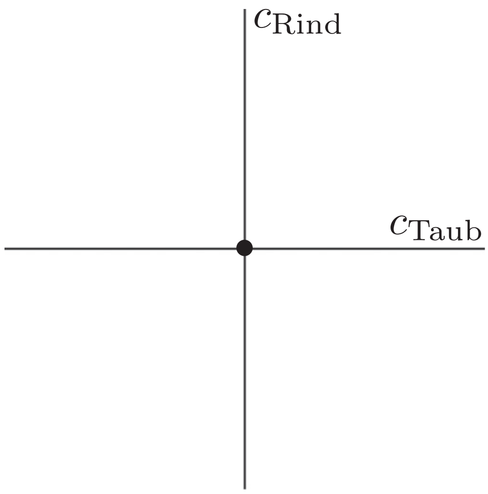

Figure 1. The one-parameter family of flat (Rindler) solutions meets the one-parameter family of curved (Taub) solutions at the origin, which is Minkowski space.

4. Vacuum solutions

4.1. Field equations

Without loss of generality, we now set $\Lambda = 0$. In this section we find all solutions of the Einstein equations outside the source. With these two simplifications the equations are

$$(\Psi')^2 + 2\Phi'\Psi' = 0, \tag{30}$$

$$\Psi'' + \frac{3}{2}(\Psi')^2 = 0, \tag{31}$$

$$\Phi'' + (\Phi')^2 + 2\Phi'\Psi' = 0. \tag{32}$$

The constraint equation (30) requires either $\Psi' = 0$ or $\Psi' = -2\Phi'$.

4.2. Solutions with $\Psi' = 0$

In this case (31) is vacuous and (32) becomes

$$\Phi'' + (\Phi')^2 = 0. \tag{33}$$

4.2.1. Subcase $\Phi' = 0$. Here both Ψ and Φ are constants. Thus all the coefficient functions in the metric (3) are constants, and by rescaling the coordinates we may take the constants to equal unity: The solution is Minkowski space (or a portion of it) in Cartesian coordinates.

4.2.2. Subcase $\Phi' \neq 0$. The equation (33) is separable, with solution

$$\Phi' = \frac{1}{z - c}$$

for some constant c. It follows that

$$e^{2\Phi} = a^2(z - c)^2, \qquad e^{2\Psi} = b^2.$$

The metric is now

$$ds^2 = -a^2(z - c)^2\, dt + b^2\left(dx^2 + dy^2\right) + dz^2, \tag{34}$$

which is recognizable (e.g., [8]) as a form of the Rindler metric [9] with horizon at $z = c$. Famously, (34) describes a piece of space-time that is flat but 'seen' from the point of view of a uniformly accelerated observer at constant z.

One could eliminate the three constants of integration by redefining z by an additive constant and the other three coordinates by multiplicative constants, thereby getting a standard form of the Rindler metric (e.g., [10, (2.35)]). We prefer, however, to leave c undetermined, because it describes how the coordinate system is situated in space relative to the slab (assumed to be at $z = 0$ by convention). To understand the full physical significance of this solution we shall need to construct a global solution through the slab and both sides of it, and in this task the freedom to adjust a and b will also be necessary. For now we merely note that the metric function $(z - c)^2$ is nonsingular on the side of the slab where z and c have opposite signs.

4.3. Solutions with $\Psi' = -2\Phi'$

In this case (31) and (32) become

$$\Psi'' + \frac{3}{2}(\Psi')^2 = 0, \tag{35}$$

$$\Phi'' - 3(\Phi')^2 = 0. \tag{36}$$

It is again routine to find

$$\Phi' = -\frac{1}{3}\left(\frac{1}{z - c}\right), \qquad \Psi' = \frac{2}{3}\left(\frac{1}{z - c}\right),$$

and the constraint equation forces the two constants c to be the same. After a few more standard steps, the final form of the metric becomes

$$\begin{aligned} ds^2 = a^2\frac{1}{\sqrt[3]{(z - c)^2}}dt^2 \\ + b^2\sqrt[3]{(z - c)^4}\left(dx^2 + dy^2\right) + dz^2. \end{aligned} \tag{37}$$

This time the singularity at $z = c$ is a genuine curvature singularity. The full Riemann tensor (11)–(17) does not vanish, and with $\Lambda = 0$ one calculates

$$\begin{aligned} R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = 4\Big[2\Phi'^2\Psi'^2 + \Phi''^2 + 2\Phi''\Phi'^2 + \Phi'^4 \\ + 2\Psi''^2 + 4\Psi''\Psi'^2 + 3\Psi'^4\Big] \end{aligned} \tag{38}$$

in general and

$$\begin{aligned} R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} &= 4\left[57\Phi'^4 + 9\Phi''^2 - 30\Phi''\Phi'^2\right] \\ &= \frac{64}{27}(z - c)^{-4} \end{aligned} \tag{39}$$

in the case at hand. Most authors who have discussed the strange solution (37), such as Muñoz and Jones [11], reject it as unphysical because of the singularity. Amundsen and Grøn [4], however, regard the singularity as real and inevitable. In our context, however, these conclusions demand reexamination. The metric expression (37) (like (34)) is not intended to be relevant for all values of the coordinates. It holds only in a certain region of space-time outside the slab. Inside the slab *and in the vacuum region on the other side* different expressions should apply. As in the Rindler case, there is no singularity (even as a coordinate artifact) so long as the c on the z-positive side of the slab is negative and the c on the z-negative side is positive. It remains to check that this behavior is indeed predicted by the boundary conditions imposed by a reasonable model of the interior of the slab, and that will be the main theme of the remainder of this paper.

The curved metric (37), often expressed in a different gauge, has been called the *Taub solution*, although (a) Levi–Civita and Kasner found it three decades before Taub [4], and (b) there is another, more famous, metric associated with the name of Taub.

Note that, because of the sign of the derivative of $g_{11} = \sqrt[3]{(z-c)^4}$, the slab appears spatially concave to a nearby observer on the side with the singularity, and convex on the other side. (The Rindler horizon has no such spatial curvature.)

4.4. Summary

The set of possible vacuum solutions with planar symmetry has the structure of two intersecting lines (figure 1), the intersection being Minkowski space. The natural coordinate on each line is c^{-1}, because $c = 0$ is not allowed but $c = \infty$ is the Minkowski limit.

5. Gravitational field of a thin plate

5.1. Electrostatic analogy

Consider a plane of charge of constant density σ at $z = 0$. By Gauss's law, in units where $\nabla \cdot \mathbf{E} = \sigma$ the total electric flux outward from a surface element must equal σ per unit area. The most symmetrical solution is that $\mathbf{E}$ points outward on each side with magnitude $\frac{\sigma}{2}$:

$$\mathbf{E} = \begin{cases} \dfrac{1}{2}\sigma\,\hat{z} & \text{for } z > 0, \\ -\dfrac{1}{2}\sigma\hat{z} & \text{for } z < 0. \end{cases} \tag{40}$$

To this we could add any global solution of the homogeneous equation, but most of those would spoil the translational and rotational symmetries. Only the reflection symmetry is ruined, however, by adding a constant vector field. In particular, a perfectly good solution is

$$\mathbf{E} = \begin{cases} \sigma\,\hat{z} & \text{for } z > 0, \\ 0 & \text{for } z < 0. \end{cases} \tag{41}$$

Both (40) and (41) are legitimate solutions, and it is hard to argue that one is more physical than the other. For a sphere of charge this ambiguity does not exist: spherical symmetry and the boundary condition of regularity at the origin force $\mathbf{E}$ to be zero inside the sphere and outward of magnitude σ outside. If we sit at a point on the sphere and send the radius to infinity, we approach a scenario of type (41). On the other hand, if we have two parallel planes of the type (41), with zero electric field between, and bring them together, we obtain a plane of type (40) with charge density 2σ.

The conclusion is that a decision between the two solutions cannot be based dogmatically on some abstract asymptotic condition. It depends on how the charge sheet is situated in a broader scheme of things—usually being an idealized model of a spatially bounded charge distribution that may coexist with other charges, conductors, etc at a distance. It is no surprise that similar issues arise in the gravitational problem, but there they are complicated by the nonlinearity of the equations.

5.2. Field equations

Consider now a matter source confined to a plane. In dealing with this idealized problem we shall proceed somewhat pontifically, expecting physical justifications to evolve later, in a limit, from an understanding of similar problems with less singular matter distributions. In problems of such high symmetry it is not necessary to use the full theory of thin shells in terms of the second fundamental form [12–14]. In all forms of the metric we have constructed, z is the physical distance of a point from the plane. Therefore, although different metric expressions apply on the two sides of the plate, it is meaningful to directly compare z-derivatives there.

We postulate the two field equations

$$\Psi'' + \frac{3}{2}\Psi'^2 = -4\pi\rho_0\delta(z), \tag{42}$$

$$\Phi'' + \Phi'^2 + \Psi'\Phi' - \frac{1}{2}\Psi'^2 = 4\pi\left(\rho_0 + 2p_0\right)\delta(z), \tag{43}$$

with constants ρ_0 and p_0 , and we interpret them in the standard way: Ψ and Φ are continuous,

$$\Psi\left(0^+\right) = \Psi\left(0^-\right), \qquad \Phi\left(0^+\right) = \Phi\left(0^-\right), \tag{44}$$

but their derivatives have finite jumps,

$$\Psi'\left(0^+\right) - \Psi'\left(0^-\right) = -4\pi\rho_0, \tag{45}$$

$$\Phi'\left(0^+\right) - \Phi'\left(0^-\right) = 4\pi\left(\rho_0 + 2p_0\right). \tag{46}$$

It follows that p_z in the constraint equation (27) cannot contain a delta function (unless we model the other components with distributions that are even more singular), and hence the third equation is just

$$\Psi'^2 + 2\Phi'\Psi' = 0. \tag{47}$$

5.3. Matching

Instead of postulating values for energy density and pressure on physical grounds, we shall identify two different vacuum solutions along the plane $z = 0$ and see what values are thereby predicted by (45) and (46). The conditions (44) and (47) must be imposed. Recall that there are three classes of solutions, Minkowski space in the general form

$$ds^2 = -a^2\, dt^2 + b^2\left(dx^2 + dy^2\right) + dz^2,$$

for which

$$\Psi'(0) = 0, \quad \Phi'(0) = 0, \quad e^{2\Psi(0)} = b^2, \quad e^{2\Phi(0)} = a^2; \tag{48}$$

Rindler solutions (34),

$$\Psi'(0) = 0, \quad \Phi'(0) = -\frac{1}{c},$$
$$e^{2\Psi(0)} = b^2, \quad e^{2\Phi(0)} = a^2c^2; \tag{49}$$

and curved solutions (37),

$$\Psi'(0) = -\frac{2}{3c}, \quad \Phi'(0) = \frac{1}{3c},$$
$$e^{2\Psi(0)} = b^2c^{4/3}, \quad e^{2\Phi(0)} = a^2c^{-2/3}. \tag{50}$$

Parameters for the solution on the right ($z > 0$) will be indicated by a subscript '+', those on the left by '−'. In all the cases, (47) is satisfied by construction and need not be mentioned further.

5.4. Both sides Rindler or Minkowski

Suppose that the left half-space is Minkowski. Then if the right side is also, the only solution is the trivial one with source zero. If the right half-space is Rindler, the solutions of (45) and (46) are

$$\rho_0 = 0, \qquad p_0 = -\frac{1}{8\pi c_+}. \tag{51}$$

Then (44) is satisfied by taking

$$a_- = a_+\left|c_+\right|, \qquad b_- = b_+. \tag{52}$$

If both sides are Rindler, the results are

$$\rho_0 = 0, \qquad p_0 = \frac{1}{8\pi}\left(\frac{1}{c_-} - \frac{1}{c_+}\right), \tag{53}$$

$$a_-\left|c_-\right| = a_+\left|c_+\right|, \qquad b_- = b_+. \tag{54}$$

Half-Minkowski solutions can be regarded as limits of full-Rindler solutions as $c_\pm \to \infty$.

The vanishing of the energy might be regarded as unphysical, but these solutions are significant as limits of a more general class to come. If c_+ is positive or c_- is negative, there will be a Rindler horizon on the respective side. Thus p_0 will be positive if there is no horizon. But a horizon is just a nuisance, not a disaster: if p_0 is negative, the flat space-time can be extended beyond the horizon in a Cartesian coordinate system (in which, of course, the plate is not at rest).

With regard to uniqueness, observe that for a fixed p_0, the '+' parameters can be chosen arbitrarily and the '−' parameters are then determined. There is no reason not to adopt the further convention $b = 1$, but in general one must have a nontrivial a on at least one side, if one insists on a consistent time coordinate in a full neighborhood of the plate. Finally, notice that different values of c_+, say, correspond to different physical situations, because $-\frac{1}{c_+}$ is the acceleration of the plate as observed on the positive side (when $a_+ = 1$). A different acceleration is thereby dictated on the negative side. This is not a paradox: One must not think of the plate as travelling through a single Minkowski space; rather, fragments of two different spaces are joined at the worldsheet of the plate, and the space-time as a whole is not flat. Acceleration is a statement about the *extrinsic* curvature of the worldsheet; its internal geometry is flat (3-dimensional Lorentzian). The physically different geometries parametrized by c_+ clearly are analogous to the different solutions of the electrostatic problem parametrized by the magnitude of the electric field in the positive region. See figure 2.

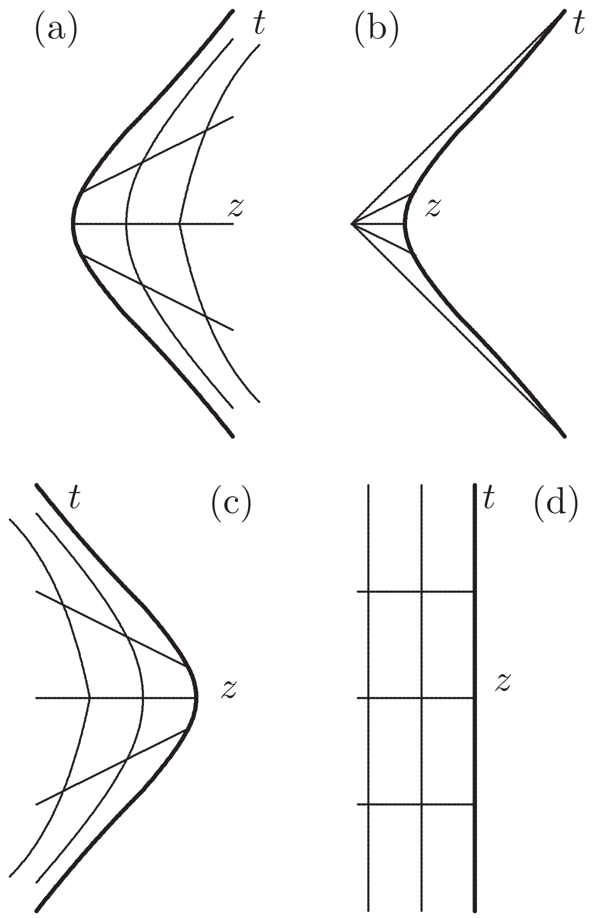

Figure 2. (a) Rindler-like coordinate system to the right of the plate, for $c_+ < 0$. (b)–(d) The situation on the left of the plate may be like any of these, depending on the sign of c_-.

If $c_+ = -c_-$, the geometry is symmetric under the reflection $z \to -z$. That solution (including the necessary condition of vanishing energy) was found by Horský [15].

5.5. Both sides curved

Let the geometry on each side of the plate be described by a metric of the form (37). From (50) we have

$$\rho_0 = \frac{1}{6\pi}\left(\frac{1}{c_+} - \frac{1}{c_-}\right), \qquad \rho_0 + 2p_0 = \frac{1}{12\pi}\left(\frac{1}{c_+} - \frac{1}{c_-}\right), \tag{55}$$

$$a_- \left|c_-\right|^{-1/3} = a_+ \left|c_+\right|^{-1/3}, \qquad b_- \left|c_-\right|^{2/3} = b_+ \left|c_+\right|^{2/3}. \tag{56}$$

Thus all solutions of this class have

$$p_0 = -\frac{1}{4}\rho_0. \tag{57}$$

If the signs of $c_\pm$ are such that the geometry is nonsingular, then $\rho_0 < 0$ (with empty Minkowski space as limiting case); if the energy is negative, then nonsingular solutions exist, though singular solutions with $c_\pm$ of the same sign are also possible. Again the multiplicative constants can be normalized to unity on one side, and again one of the additive constants $c_\pm$ parametrizes a family of distinct solutions differing in the location and extrinsic curvature of the plate's worldsheet, $c_+ = -c_-$ being a reflection-symmetric choice.

The surprising (but previously known [14]) result (57) shows that this class of solutions is just as constrained as the class with both sides flat, where the analog is $\rho_0 = 0$. The algebraic reason is that one linear combination of (42) and (43) is

$$4\pi\left(\rho_0 + 4p_0\right)\delta(z) = (2\Phi' + \Psi')' + \frac{1}{2}(2\Phi' + \Psi')^2, \tag{58}$$

and since $2\Phi' + \Psi' = 0$ on both curved vacuum sides, the quantity $\rho_0 + 4p_0$ cannot jump at $z = 0$.

Note that if one were to study a problem with *two parallel plates*, the space-time between them could be of type (37). From the point of view of one of the plates, c would have the 'wrong' sign, but the solution would not have a singularity. We cannot denounce a solution as singular until we know what lies beyond it—a vacuum extension with an unavoidable singularity, or another matter source that ends the solution's relevance. We shall appeal to this principle again in section 5.7.

5.6. The mixed case

If the left side is Rindler and the right side is curved, we get

$$\rho_0 = \frac{1}{6\pi c_+}, \qquad \rho_0 + 2p_0 = \frac{1}{4\pi}\left(\frac{1}{3c_+} + \frac{1}{c_-}\right), \tag{59}$$

$$a_- \left|c_-\right| = a_+ \left|c_+\right|^{-1/3}, \qquad b_- \left|c_-\right| = b_+ \left|c_+\right|^{2/3}. \tag{60}$$

At last we have energy and pressure that are independent:

$$p_0 = \frac{1}{8\pi}\left(\frac{1}{c_-} - \frac{1}{3c_+}\right) = -\frac{1}{4}\rho_0 + \frac{1}{8\pi c_-}. \tag{61}$$

Positivity of ρ_0 is equivalent to positivity of c_+, which is the condition for the curved solution to have a singularity. Positivity of c_-, the condition that the coordinate system on the flat side has no horizon, is equivalent to

$$p_0 \geqslant -\frac{1}{4}\rho_0. \tag{62}$$

(Equality corresponds to $c_- = \infty$, the case of Cartesian Minkowski space on the left and the curved solution on the right.) But as previously discussed, the existence of a horizon is not a problem.

If ρ_0 and p_0 are given, the equations (59) determine c_+ and c_- (which may take the value ∞ but not 0). Unlike in the previous cases, there is no freedom to choose the location of the plate within the space-time on one side. There is only the discrete freedom to put the flat side on the right instead of the left. In particular, *unless* $\rho_0 = 0$ *or* $p_0 = -\frac{1}{4}\rho_0$, *there is no reflection-symmetric solution for a source of the form assumed in (42)–(43).*

The most novel feature of this paper is the construction of solutions that are not reflection-symmetric. All previous authors seem to assume, tacitly or explicitly, that a solution with vacuum on both sides of a plane must be symmetric. On the other hand, other authors have weakened some of our other conditions, constructing solutions with time dependence ([16] and many later papers), anisotropy in the x–y plane [17], a cosmological constant [8, 18], or nonvacuum media outside the slab [18, 19].

5.7. Why does the energy not want to be positive?

We have found one class of solutions in which ρ_0 is 0 and two other classes in which negativity of ρ_0 is required to avoid a curvature singularity at some distance from the plate. The conclusion that positive energy density forces a singularity was also obtained in [4] by a different argument. How can this be, when positivity is physically expected for normal matter?

Consider the infinite plane as the limit of a disk of radius R. As the radius increases with the density constant, the total mass increases as R^2, or $R \propto \sqrt{m}$. As an astrophysical object, the disk has a Schwarzschild radius $R_S \propto m$; eventually R_S must dominate. (To forestall any qualms over how meaningful the Schwarzschild radius is for a flat disk, one can think of the surface of the slab as the limit of the surface of a spherical shell.) This indicates that our basic physical picture is wrong for a plate that is too big: like an overly massive star, an infinite massive plate can't exist as a static object. We must expect that the model makes sense only if ρ_0 approaches 0 as $R \to \infty$.

On the other hand, we have grown up with Galilean gravity, where the surface of the Earth is treated as an infinite plane exerting a uniform gravitational force. This empirically verified picture is modified only slightly by either the curvature of the Earth or the gravitational redshift (meaning in this context that the various Rindler hyperbolas in figure 2 have different curvatures). Certainly it must be obtainable in some limit from an exact general-relativistic treatment. Consider a small region of space very close to a moderately massive sphere or disk, and close to the center in the disk case. 'Moderately massive' means that the body is safely larger than its Schwarzschild radius, although its mass is large enough to make general relativity relevant. Although we do not have the time and space here to develop the idea quantitatively, we believe that in this small region the asymptotic form of the exact solution of the problem must be identifiable with one of the slab solutions, likely one in the curved class, (37), with $c_+ > 0$. The explanation of the curvature singularity is then that it occurs so far away from the plate that the metric is no longer a good approximation to the exact one (e.g., a distance comparable to the radius of the massive body). A further Newtonian limit must then restore the Galilean picture, the nontrivial metric coefficient g_{11} becoming unimportant because of the largeness of the speed of light. (Similar remarks have been made by Lemos and Ventura [20].)

6. Slabs of finite thickness

Even aside from the problem of the sign of the energy, in the previous section we encountered some unexpected restrictions, which suggest that some physically meaningful solutions are being lost by passing to the idealized plate limit—probably because of the restriction to vanishing p_z. In any case, solutions with distributional sources are somewhat suspect until shown to be limits of solutions with nonsingular sources. We therefore begin the study of extended slabs.

6.1. Equations

In general, the three field equations, in the gauge $\Lambda = 0$, are

$$\Psi'' + \frac{3}{2}\Psi'^2 = -4\pi\rho, \tag{63}$$

$$\Phi'' + \Phi'^2 + \Psi'\Phi' = \frac{1}{2}\Psi'^2 + 4\pi(\rho + 2p), \tag{64}$$

$$8\pi p_z = \Psi'(\Psi' + 2\Phi'). \tag{65}$$

There are two philosophies one could adopt toward equation (65) and toward the entire issue of prescribing a matter source.

First, one could solve (63) and (64) for given functions $\rho(z)$ and $p(z)$ and then accept whatever function $p_z(z)$ is determined by (65). An even more extreme option is to postulate functions $\Psi'(z)$ and $\Phi'(z)$ and then accept the stress tensor determined by the three equations. Our treatment of the plate in the previous section was a combination of these two approaches. One must be prepared to encounter energies and pressures that are physically implausible.

Second, one could insist that ρ, p, and even p_z must be determined by local physics—an equation of state, or some justifiable generalization thereof. Then (65) is a *constraint* on the initial data, $\Psi'(0)$, $\Phi'(0)$, and $p_z(0)$. This constraint will then automatically also hold at other values of z, because the conservation law (26) must be satisfied for any stress tensor consistent with conservation of energy.

Unfortunately, it is not clear how restrictive one can allow 'local physics' to be in such a highly anisotropic scenario as we have committed to in this project. Cosmologically, normal matter is unlikely to condense into a stable slab. A useful analogue is the cylindrical scenario reviewed in [3]. One can construct solutions where the cylindrical core is a traditional fluid with an isotropic equation of state $p_z = p_r = p_\theta = \omega\rho$, typically with $0 \leqslant \omega \leqslant \frac{1}{3}$, but they are probably unstable against condensation into 'beads' along the axis. (For a recent discussion of this issue see [21].) Mathematically more natural are various solutions with anisotropic pressures. In particular, the most studied cases are 'cosmic strings' with $p_r = p_\theta = 0$ and $p_z = -\rho$ (which are invariant under Lorentz transformations along the cylinder axis). Although implausible for normal matter, this kind of source is natural in certain nonAbelian gauge theories [21].

This situation gives us an excuse to retreat, at least on this occasion, to the first point of view, which is easier to implement calculationally. Indeed, we will sometimes be forced to resort to its extreme version, which is *much* easier as it involves only differentiating functions, not solving differential equations.

Note that the field equations are *first-order* differential equations for Ψ' and Φ'. Integrating to find Ψ and Φ is a trivial afterthought. Furthermore, there is no loss of generality in setting

$$\Psi(0) = 0 = \Phi(0); \tag{66}$$

this is a normalization condition that recognizes the restriction of the metric (3) to the surface $z = 0$ to be three-dimensional special-relativistic flat space, hereby put into standard Cartesian form.

6.2. Solutions

First we observe that if $p_z = 0$ everywhere, then either ρ is identically zero (by (63)) or $p = -\frac{1}{4}\rho$ (by (58) generalized to the extended source). On the other hand, if the slab has a surface, say at $z = \pm\epsilon$, outside which the stress tensor is zero, then a solution must have $p_z(\pm\epsilon) = 0$ when the surface is approached from the inside also, since we do not wish now to consider distributional ρ and p. This is also a physical requirement for a static solution: a normal pressure on the slab boundary, uncompensated by a pressure from outside, would cause the boundary to accelerate.

6.2.1. Symmetric solutions To narrow the field let us now concentrate on solutions that are reflection-symmetric in z. As a step toward physical realism, we shall assume that ρ is nonnegative (and not identically 0). Then (63) forces Ψ' to be decreasing, and the reflection symmetry forces $\Psi'(0) = 0$. Although (64) does not force Φ'' to have a particular sign, most of the terms tend to make it positive. In fact, if we are to have reflection symmetry and also $p_z(\pm\epsilon) = 0$, we must have

$$2\Phi'(\pm\epsilon) = -\Psi'(\pm\epsilon) \quad \text{and} \quad \Phi'(0) = 0. \tag{67}$$

Note that $p_z(0) = 0$ also. To show the existence of interesting solutions we must satisfy these conditions without having $p_z = 0$ everywhere. The adjoined vacuum solutions will be of the type (37), with singularities by virtue of (50) and the sign conditions just mentioned, and will be mirror images of each other.

Even when ρ is constant, (63) can't be solved analytically. Therefore, for the purpose of producing a solution with reasonable, if somewhat arbitrary, structure, we shall simply postulate that $\Psi'(z) = -bz$ for some positive constant b when $|z| < \epsilon$. Then, according to (63),

$$4\pi\rho = b - \frac{3}{2}b^2z^2, \tag{68}$$

which is positive if $b < 2/3\epsilon$. Substituting into (64) we get

$$\Phi'' + \Phi'^2 - bz\Phi' = b - b^2z^2 + 8\pi p. \tag{69}$$

Let us make the ansatz

$$\Phi'(z) = \frac{1}{2}bz + q(z), \tag{70}$$

$$q(z) = az(z-\epsilon)(z+\epsilon) = a\left(z^3 - \epsilon^2 z\right), \tag{71}$$

which satisfies (67). Then

$$8\pi p = a^2z^6 - 2a^2\epsilon^2z^4 + \left(3\,a + \frac{3}{4}b^2 + a^2\epsilon^4\right)z^2 - \left(\frac{b}{2} + a\epsilon^2\right) \tag{72}$$

and

$$8\pi p_z = -2bzq(z). \tag{73}$$

Note that $p_z \geqslant 0$ if $a > 0$, but then $p(0) < 0$ and if a and ϵb are sufficiently large p changes sign inside the slab:

$$8\pi p(\epsilon) = \frac{8a}{9b^2} - \frac{b}{2} + \frac{1}{3} \quad \text{if } \epsilon = \frac{2}{3b}.$$

This pressure function has no physical foundation, but it demonstrates that symmetrical slab systems with $p \neq \frac{1}{4}\rho$ do exist mathematically.

A better approach might be to prescribe physically reasonable formulas for ρ and p and then to solve the field equations numerically. Here we merely point out that such a venture is not guaranteed success. In analogous problems with spherical [1] and cylindrical [3] symmetry, the quantity analogous to p_z is the radial pressure, p_r . In those cases, $p_r(0)$ can (and must) be prescribed an arbitrary positive value, and then the conservation law analogous to (26) drives p_z monotonically to 0 at a value of z that can (and must) be interpreted as ϵ, the boundary of the body. In the present problem, with reflection symmetry, the initial value of p_z must be 0, and it is not clear that the numerical solution for p_z will ever reach 0 elsewhere. Certainly a nontrivial p_z with three zeros cannot be monotonic.

The oscillatory structure of (72) and (73) suggests limiting forms (as $\epsilon \to 0$) containing $\delta''(z)$, in keeping with our previous speculation. We have not investigated this possibility further.

6.2.2. Pressureless solutions. Recall that our original motivation was a slab of ordinary metal in a laboratory studying vacuum energy. This is an effectively nonrelativistic system, and one would expect the pressure inside it to be much less than the energy density. Therefore, a better approach than the foregoing may be to assume $p = 0$ as the equation of state of the tangential pressure. The equation analogous to (58) is now

$$\Omega'' + \frac{1}{2}\Omega'^2 = 4\pi\rho, \qquad \Omega \equiv 2\Phi + \Psi. \tag{74}$$

This equation can be used to replace (64), so our equation set is now (63), (65), and (74) .

Given a nonnegative function ρ on the interval $[-\epsilon, \epsilon]$, one can numerically integrate (63) to get a decreasing function Ψ' satisfying $\Psi'(-\epsilon) = 0$. Then one can integrate (74) with the data $\Omega'(+\epsilon) = 0$. At least initially, Ω' will be increasing (i.e., negative in the interval to the left of $z = \epsilon$). By construction, $p_z(\pm\epsilon) = 0$. Inside the interval, p_z is nonzero (probably positive), but if the slab is fairly thin, p_z will never be large compared to ρ. (In conventional units, Ψ' and Ω' are $O(G)$ when $\rho = O(1)$, and moreover their product is $O(z \pm \epsilon)$.) If the slab is thick, the pressure in the center might well become 'relativistically' large because of gravitational compression, as in a massive star.

Any of these solutions will join on to a Rindler solution on the left and a curved solution on the right, but without the constraints found in section 5.6 on the locations of the embedded boundaries. The z-reversed solution is obtained by interchanging the roles of $\pm\epsilon$ in the construction.

6.3. Domain wall solutions

In theories with spontaneous symmetry breaking, thin boundaries can appear between different phases. As for cosmic strings, the expected equation of state is $p = -\rho$, and then a plane boundary will be invariant under Lorentz transformations in the t–x–y space. Cosmologically relevant solutions are expected to be time-dependent, however. We cannot review, much less compete with, the literature on this topic, so we cite just one paper containing many references [18].

7. Conclusions

Among previous papers the one we have found most valuable is that of Amundsen and Grøn [4]. It contains a lengthy bibliography of earlier work. (In contrast, for expository reasons we have chosen to concentrate on recent references.) More importantly, [4] gives by far the best treatment of the gauge freedom in the problem. We somewhat differ with those authors' physical interpretation, however. Their paper does not discuss the geometry and stress tensor of the source nor the geometry on the 'other side' (if there is one) of the source. Instead, they define the energy density of the source from the acceleration of test particles. They thereby conclude that positive energy requires a curved solution (37) with a singularity in the physical region. We have speculated that the singularity must appear in a region where the infinite-slab model has already broken down for other reasons.

In this paper we have investigated in full detail the most general solution associated with a source localized on a plane with no behavior more singular than a Dirac delta function. For two very special equations of state we find one-parameter families of solutions very analogous to those of the electrostatic problem of a sheet of charge, including one solution that is reflection-symmetric and others where the two sides are qualitatively the same (both flat or both curved). But in general, a flat solution on one side dictates a curved one on the other and vice versa, and the one-parameter freedom in the plate's location is lost.

We attribute this strange result to the impossibility of having nonzero pressure normal to the plate in these models. Therefore, we began the study of more general models with extended slab sources. More general pairs of external geometries thereby become possible, but the situation is still hard to understand physically. For example, reflection-symmetric solutions require stress tensors that are either physically implausible or rather contrived. Further investigation will require numerical calculations beyond the scope of this paper.

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