

GLOBAL AND LOCAL VACUUM ENERGY AND CLOSED ORBIT THEORY

S. A. FULLING

Mathematics Dept., Texas A&M University, College Station, TX 77843-3368, USA

E-mail: fulling@math.tamu.edu

The report has two themes: the relation between Casimir energy and the analysis of spectral oscillations by periodic orbits, and the renormalization of energy densities near boundaries and the issue of their consistency with renormalized total energies.

1 The Twisted Periodic Model

1.1 The Facts

We¹ recently considered a massless scalar field in one spatial dimension satisfying the boundary condition $\phi(t, L) = e^{i\theta} \phi(t, 0)$ and similarly for $\frac{\partial \phi}{\partial t}$, where θ is a fixed angle; that is, the field takes values in a line bundle over S^1 . The normal modes have wave numbers $(2\pi n + \theta)/L$ ($n \in \mathbf{Z}$) and hence two sequences of eigenfrequencies, $\omega_j^\pm = (2\pi j \pm \theta)/L$ ($j \in \mathbf{N}$ or \mathbf{Z}^+). The vacuum energy is

$$E_\theta = -\frac{\pi}{12L} \left[2 - 6\frac{\theta}{\pi} + 3 \left(\frac{\theta}{\pi} \right)^2 \right]. \quad (1)$$

It is noteworthy that the extreme values, $E_0 = -\frac{\pi}{6L}$ and $E_\pi = +\frac{\pi}{12L}$, come when the two sequences coincide and hence the spectrum exhibits the largest gaps, while $E_\theta = 0$ for $\theta \approx 0.42\pi$, close to the value $\theta = \frac{\pi}{2}$ where the full spectrum is equally spaced. Initially this and similar observations suggested that Casimir energy should be suppressed in chaotic systems, where eigenvalues are less clumpy than in integrable ones. However, closer inspection of the calculation shows that the result owes more to the *phases* of the individual eigenvalue sequences than to their relative position. My current belief, therefore, is that attention to eigenvalue oscillations (especially their locations along the frequency axis) will be most valuable in connection with the notorious question of the *sign* of vacuum energy in various configurations.

1.2 Analysis by Periodic Orbits

In the original paper¹ (1) in effect was derived in two ways, directly from the eigenvalue spectrum and from the asymptotics of the cylinder kernel (see below). Here we outline a third derivation, more in keeping with the modern literature of quantum mechanics. We construct the nonrelativistic quantum Green function (solving $(H - E)G(E, x, y) = \delta(x - y)$, with $H = -d^2/dx^2$) from the periodic orbits of the related classical system. In this simple model

this construction reduces to the elementary “method of images”. The Green function for infinite space is $G_\infty(\omega^2, x, y) = \frac{i}{2\omega} e^{i\omega|x-y|}$. That for the twisted cylinder is therefore

$$G_{L,\theta}(\omega^2, x, y) = \sum_{n=-\infty}^{\infty} G_\infty(\omega^2, x, y + nL) e^{in\theta}. \quad (2)$$

It follows that the level density is

$$\begin{aligned} \rho(\omega) &= \sum_j \delta(\omega - \omega_j) = \frac{2\omega}{\pi} \text{Im Tr} G(\omega^2) \\ &= \frac{2\omega L}{\pi} \text{Im} G(\omega^2, x, x) = \frac{iL}{\pi} \text{Im} \sum_{n=-\infty}^{\infty} e^{i\omega|n|L} e^{in\theta}. \end{aligned} \quad (3)$$

In the first exponent one recognizes the *action* of the periodic orbit that circles the universe n times; in the second, θ plays a role analogous to the famous *Maslov index*, which does not occur in this model.

Separating the term from $n = 0$, we have

$$\begin{aligned} \rho(\omega) &= \frac{L}{\pi} + \frac{L}{\pi} \sum_{n=1}^{\infty} [\cos(\omega n\pi + n\theta) + \cos(\omega n\pi - n\theta)] \\ &\equiv \rho_{\text{av}}(\omega) + \rho_{\text{osc}}(\omega). \end{aligned} \quad (4)$$

The *average* term of the eigenvalue density contributes to the energy precisely the vacuum energy of infinite space, which is removed by renormalization. The *oscillatory* term gives rise to the Casimir energy as follows. Define a *regularized* energy by

$$E(t) \equiv \int_0^\infty \frac{1}{2} \omega \rho(\omega) e^{-\omega t} d\omega = \sum_j \frac{1}{2} \omega_j e^{-\omega_j t}. \quad (5)$$

As $t \rightarrow 0$ the divergent term is exactly the ρ_{av} term, and the contribution of ρ_{osc} is

$$E_{\text{osc}}(t) = -\frac{L}{\pi} \sum_{n=1}^{\infty} \frac{-t^2 + (nL)^2}{[t^2 + (nL)^2]^2} \cos(n\theta), \quad (6)$$

whose limit is the renormalized energy

$$E_{\text{osc}}(0) = -\frac{L}{\pi} \sum_{n=1}^{\infty} \frac{1}{(nL)^2} \cos(n\theta) = -\frac{\pi}{L} B_2 \left(\frac{\theta}{2\pi} \right), \quad (7)$$

where B_2 is a Bernoulli polynomial. Equations (1) and (7) agree.

2 Periodic Orbit Theory

The relation between eigenvalues and periodic orbits traces back to the 19th-century Poisson summation formula (which suffices to treat the simple one-dimensional models in this talk). It has been extended to surfaces of constant negative curvature,² cavities,³ manifolds,⁴ chaotic potentials,⁵ integrable potentials,⁶ and potentials of intermediate symmetry,⁷ and is now well known in atomic, nuclear, molecular, and condensed-matter physics. Its connection with Casimir energy is at least implicit in the work of Balian and Duplantier⁸ and has been in the backs of the minds of many people in the field,^{9,10,11} but I believe that Schaden and Spruch¹² were the first to use periodic orbits as a serious tool for the calculation of vacuum energies. (See also Mazzitelli et al.¹³ and the contribution of Mazzitelli to this volume.)

Here I summarize the facts catalogued in a number of recent papers^{14,1,15} about the relationships among the vacuum energy, the integral kernels associated with the differential operator in question, and the eigenvalue density (or comparable local quantities for energy density calculations and/or systems with continuous spectra).

First it is necessary to define the *Riesz means* of the eigenvalue sequence. Let $N(\lambda)$ be the number of eigenvalues ω_n^2 of the operator H that are less than or equal to λ . Note in passing that the regularized energy density (5) can be written as a Stieltjes integral

$$E(t) = -\frac{1}{2} \frac{d}{dt} \sum_n e^{-\omega_n t} = -\frac{1}{2} \frac{d}{dt} \int e^{-t\sqrt{\lambda}} dN(\lambda). \quad (8)$$

Now define the Riesz mean of order α with respect to λ as

$$R_\lambda^\alpha N(\lambda) = \frac{1}{\alpha!} \lambda^{-\alpha} \int^{\cdot\alpha} \int^\lambda N(\tilde{\lambda}) d\tilde{\lambda}, \quad (9)$$

where α successive indefinite integrations are intended. (That is, $R_\lambda^\alpha N$ is an average of N over an α -dimensional simplex). Similarly, the Riesz means with respect to the variable $\omega = \sqrt{\lambda}$ are defined by

$$R_\omega^\alpha N(\omega) = \frac{1}{\alpha!} \omega^{-\alpha} \int^{\cdot\alpha} \int^\omega N(\tilde{\omega}^2) d\tilde{\omega}. \quad (10)$$

The important equations of the subject fall into three groups:

(1) N itself does not have a literal asymptotic expansion at high frequencies beyond the famous first term of Weyl, precisely because of the spectral oscillations that are the center of our attention and the source of the Casimir energy. Each integration, however, averages out the oscillations by one degree to yield a rigorous asymptotic formula with one more term. Thus one has, as $\lambda \rightarrow +\infty$,

$$R_\lambda^\alpha N = \sum_{s=0}^{\alpha} a_{\alpha s} \lambda^{(d-s)/2} + O(\lambda^{(d-\alpha-1)/2}), \quad (11)$$

$$R_\omega^\alpha N = \sum_{s=0}^{\alpha} c_{\alpha s} \omega^{d-s} + \sum_{\substack{s=d+1 \\ s-d \text{ odd}}}^{\alpha} d_{\alpha s} \omega^{d-s} \ln \omega + O(\omega^{d-\alpha-1} \ln \omega). \quad (12)$$

The coefficients a , c , d are determined by the operator H and hence ultimately by the mechanical system whose Hamiltonian it is (including, in particular, the geometry of the region, M).

(2) The famous *heat kernel expansion* says that the Green function $K(t, x, y) \equiv \langle x | e^{-tH} | y \rangle$ has the small- t behavior

$$\sum_{n=1}^{\infty} e^{-t\omega_n^2} = \int_M K(t, x, x) dx \sim \sum_{s=0}^{\infty} b_s t^{(-d+s)/2}. \quad (13)$$

The heat-kernel coefficients are related to the Riesz-mean coefficients by

$$b_s = \frac{\Gamma((d+s)/2+1)}{\Gamma(s+1)} a_{ss}. \quad (14)$$

More pertinent to Casimir calculations is the *cylinder (Poisson) kernel*, $T(t, x, y) \equiv \langle x | e^{-t\sqrt{H}} | y \rangle$ (which solves a boundary-value problem for the elliptic equation $\frac{\partial^2 u}{\partial t^2} - Hu = 0$ in the space $\mathbf{R}^+ \times M$ with data on the face M). Its expansion as $t \rightarrow 0$ is

$$\sum_{n=1}^{\infty} e^{-t\omega_n} = \int_M T(t, x, x) dx \sim \sum_{s=0}^{\infty} e_s t^{-d+s} + \sum_{\substack{s=d+1 \\ s-d \text{ odd}}}^{\infty} f_s t^{-d+s} \ln t, \quad (15)$$

where the coefficients are related to those in (12) by

$$e_s = \frac{\Gamma(d+1)}{\Gamma(s+1)} c_{ss} \quad \text{if } d-s \text{ is even or positive}; \quad (16)$$

$$f_s = -\frac{\Gamma(d+1)}{\Gamma(s+1)} d_{ss}, \quad e_s = \frac{\Gamma(d+1)}{\Gamma(s+1)} [e_{ss} + \psi(d+1)d_{ss}] \quad (17)$$

if $d-s$ is odd and negative.

(3) So far we have discussed two sets of quantities in parallel. They are related this way:

$$c_{ss} = \frac{\Gamma((d-s+1)/2)\Gamma((d+s)/2+1)}{2^s\Gamma((d+1)/2)\Gamma(d/2+1)} a_{ss} \quad \text{if } d-s \text{ is even or positive}, \quad (18)$$

whereas if $d-s$ is odd and negative

$$d_{ss} = \frac{(-1)^{d+1}\Gamma((s-d)/2)}{(s-d-1)!d!\Gamma(-(d+s)/2)} a_{ss}, \quad \text{but } c_{ss} \text{ is undetermined by the } a_{rr}. \quad (19)$$

Combining these equations with those above for the kernel expansions, we see that the cylinder coefficients determine the heat coefficients, but not entirely vice versa:

$$e_s = \pi^{-1/2} 2^{d-s} \Gamma((d-s+1)/2) b_s \quad \text{if } d-s \text{ is even or positive,} \quad (20)$$

whereas if $d-s$ is odd and negative

$$f_s = \frac{(-1)^{(s-d+1)/2} 2^{d-s+1}}{\sqrt{\pi} \Gamma((s-d+1)/2)} b_s, \quad \text{but } e_s \text{ is undetermined by the } b_r. \quad (21)$$

The “new” coefficients, e_s with $d-s$ odd and negative, are *nonlocal* in their dependence on H and the geometry of its spatial domain, M . It is only they that are sensitive to the existence of periodic orbits and to the shape effects (often miscalled “topological”) that give rise to Casimir energy. In particular, the renormalized vacuum energy is

$$E = -\frac{1}{2} e_{1+d} + \text{possible local terms.} \quad (22)$$

(The local terms may depend on the detailed definition of the renormalization prescription, but the nonlocal term does not.) Because the cylinder kernel probes the spectrum of H and the geometry of M more deeply than the heat kernel does, it is not surprising that it is harder to calculate. Nevertheless, finding the first few terms in the expansion (15) is easier than finding the whole spectrum of eigenvalues or the whole spectrum of periodic orbits.

3 Local Energy Density

3.1 Analysis by Closed Orbits

The quantum energy density $T^{00}(x)$ is related to *closed* classical paths that are not necessarily *periodic*; that is, they start and end at x but not necessarily with the same initial and final direction. For a scalar field in 2-dimensional flat space-time the formula for T^{00} contains a free parameter:

$$T^{00} = \frac{1}{2} \left[\left(\frac{\partial \phi}{\partial t} \right)^2 + \left(\frac{\partial \phi}{\partial x} \right)^2 - 4\xi \frac{\partial}{\partial x} \left(\phi \frac{\partial \phi}{\partial x} \right) \right]. \quad (23)$$

The conformally invariant choice is $\xi = 0$, but the theoretically simplest value is¹⁵ $\xi = \frac{1}{4}$; in particular, T^{00} for that case has the most direct relation to the cylinder kernel.

For a finite interval with a Dirichlet or Neumann condition at each end, the regularized $T^{00}(x)$ can be written as a sum over all paths that start at x , bounce off the ends arbitrarily often, and return to x (cf. Fig. 1 of Schaden and Spruch¹²). The sum can be evaluated in closed form as a complicated combination of hyperbolic functions (unpublished).^a After renormalization it

^aThis calculation was aided by *Mathematica* Version 4, which also created the two figures that follow.

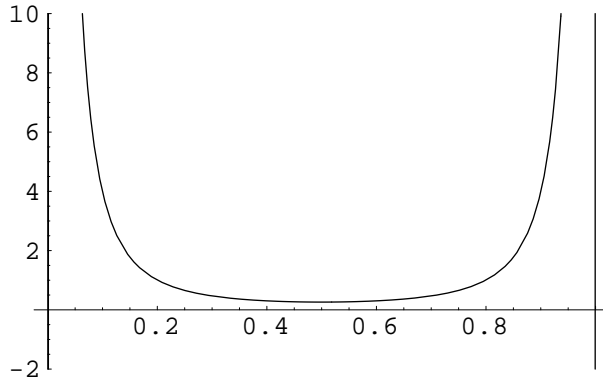


Figure 1. Vacuum energy on an interval ($\xi = \frac{1}{4}$)

exhibits Casimir energy (independent of ξ) throughout the interval, plus a term (proportional to ξ) that concentrates at the boundaries (but does not vanish at the center); see Fig. 1. The boundary behavior is more clearly seen in the simple problem of a *single* boundary at $x = 0$. With an exponential cutoff analogous to (5) one finds (for $\xi = \frac{1}{4}$)

$$T_{\text{bdry}}^{00}(t, x) = -\frac{1}{2\pi} \int_0^\infty \cos(2kx) e^{-kt} k dk = -\frac{1}{2\pi} \frac{t^2 - 4x^2}{(t^2 + 4x^2)^2}. \quad (24)$$

This function satisfies

$$\int_0^\infty T_{\text{bdry}}^{00}(t, x) dx = 0; \quad (25)$$

as long as the cutoff is in place, the boundary energy density does not contribute to the total energy (which must be independent of ξ). As the cutoff is removed, the energy of one sign immediately adjacent to the boundary (see Fig. 2) is squeezed into an increasingly narrow region and eventually disappears, leaving an uncompensated renormalized energy density of the opposite sign in a larger neighborhood of the boundary, which formally produces an infinite total energy when $t = 0$. (Similar effects have been observed earlier;^{16,17,18} see also the contribution by Graham to this volume.)

3.2 Consistency of a Model Gravitational Equation

The divergent energy density near a boundary was a matter of controversy two decades ago^{19,20,21} and remains so today (see, for instance, the contributions of Jaffe and his associates to this volume). Not only does it cast doubt on the allegedly finite renormalized total energy of finite-volume systems; it appears to be incompatible with a meaningful Einstein gravitational field equation

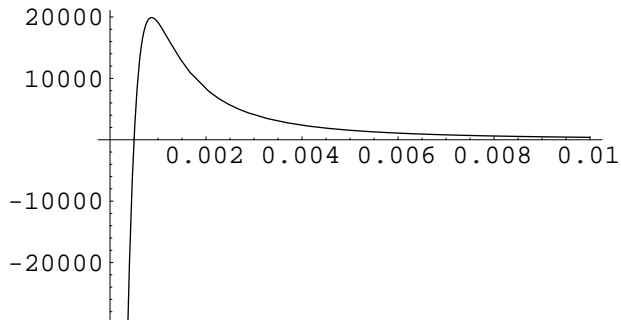


Figure 2. Closeup of the energy density near a boundary

with the local stress tensor $T^{\mu\nu}(x)$ as source. The results of the previous subsection support the old suggestion²⁰ that the nonintegrable vacuum energy can be considered to be cancelled by a singular term concentrated exactly on the boundary. Here we address the question of whether the nonintegrable term, suitably interpreted, can be treated as a source in a differential equation; this is very recent, unpublished research in collaboration with Ricardo Estrada, Klaus Kirsten, and Kimball Milton.^b

We consider a linearized 1-dimensional analog of Einstein's equation with source equal to (24) on one side only:

$$\frac{d^2y}{dx^2} = -2\pi T_{\text{bdry}}^{00}(t, x) = \frac{t^2 - 4x^2}{(t^2 + 4x^2)^2} \theta(x). \quad (26)$$

The solution is

$$y(t, x) = \frac{1}{8} \ln \left(1 + \frac{4x^2}{t^2} \right) \theta(x) + Ax + B, \quad (27)$$

where the last two terms are the general solution of the homogeneous equation, which we henceforth ignore. We take the limit $t \downarrow 0$ in both (26) and (27) in the sense of finite parts of distributions²² (discussed also by Elizalde in this volume). At small t (27) becomes

$$y \sim \frac{1}{4} \ln \left(\frac{2x}{t} \right) \equiv \frac{1}{4} \ln \left(\frac{x}{x_0} \right). \quad (28)$$

In a maneuver familiar from ordinary renormalization theory, we reinterpret the vanishing parameter $\frac{t}{2}$ as an *arbitrary but finite* parameter x_0 , which

^bOur gathering was kindly supported financially by the Baylor University Mathematics Department.

introduces a new length scale into the theory. The limit of (26) is more subtle, requiring the finite-part operation:

$$\frac{d^2 y}{dx^2} = -2\pi T_{\text{bdry}}^{00} \sim -\text{F.P.} \left(\frac{\theta(x)}{4x^2} \right) - \frac{1}{4}(1 + \ln x_0)\delta'(x). \quad (29)$$

The finite part is defined so that it integrates to 0; the operation is not scale-invariant, and that fact accounts for the apparent dimensional incoherence of the $\delta' \ln x_0$ term. (Note that the singular boundary term is proportional to δ' , not δ ; the δ contribution vanishes because of (25).) Careful attention to the definitions²² shows that (28) is indeed the solution of (29).

In other words, (26) survives the renormalization process as a mathematically consistent differential equation. I conjecture that the same thing happens to the Einstein equation with source including the notorious x^{-2} terms, proportional to boundary curvature squared, that arise in 3-dimensional electromagnetism with perfect conductors.^{19,21} Of course, only experiment can determine whether such idealized renormalized theories provide useful approximate models of laboratory systems built of real materials.

4 Conclusions

- Vacuum energy is related to the *oscillatory fine structure* of the spectral distribution, hence to *closed or periodic* classical trajectories.
- The *phases* of spectral oscillations determine the *sign* of vacuum energy.
- Cut-off energy densities are consistent with cut-off total energies (at least in the simplest models; further calculations or a more abstract analysis will be needed to confirm this conclusion in general).
- When the cutoff is removed, there appears to be a consistent renormalization procedure for using a boundary-divergent energy density as a *distributional source* in a differential equation. (This also requires further investigation.)
- Casimir-type calculations (even for unphysical models) are of mathematical interest, because they cast light on the spectral mathematics and related classical physics and geometry of partial differential operators of the Hamiltonian type. The benefits flow from physics to mathematics as well as vice versa.

References

1. S.A. Fulling, E.V. Gorbar, and C.T. Romero, Electron. J. Diff. Eqs. **Conf. 04**, 87 (2000).
2. A. Selberg, J. Indian Math. Soc. **20**, 47 (1956).

3. R. Balian and C. Bloch, *Ann. Phys. (NY)* **69**, 76 (1972).
4. Y. Colin de Verdière, *Compositio. Math.* **27**, 159 (1973).
5. M. Gutzwiller, *J. Math. Phys.* **11**, 1791 (1970); *J. Math. Phys.* **12**, 343 (1971).
6. M.V. Berry and M. Tabor, *Proc. Roy. Soc. London A* **349**, 101 (1976); *J. Phys. A* **10**, 371 (1977).
7. S.C. Creagh and R.G. Littlejohn, *Phys. Rev. A* **44**, 836 (1991); *J. Phys. A* **25**, 1643 (1992).
8. R. Balian and B. Duplantier, *Ann. Phys. (NY)* **104**, 300 (1977); *Ann. Phys. (NY)* **112**, 165 (1978).
9. P. Candelas, *Ann. Phys. (NY)* **143**, 241 (1982), Sec. 7.
10. S.A. Fulling, *J. Phys. A* **16**, 2615 (1983); *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge Univ. Press, Cambridge, 1989), p. 108.
11. M.T. Jaekel and S. Reynaud, *J. Phys. I (France)* **1**, 1395 (1991).
12. M. Schaden and L. Spruch, *Phys. Rev. A* **58**, 935 (1998); *Phys. Rev. Lett.* **84**, 459 (2000).
13. F.D. Mazzitelli, M.J. Sánchez, N.N. Scoccola, and J. von Stecher, *Phys. Rev. A* **67**, 013807 (2003).
14. S.A. Fulling and R.A. Gustafson, *Electron. J. Diff. Eqs.* **1999**, No. 6 (1999); R. Estrada and S.A. Fulling, *Electron. J. Diff. Eqs.* **1999**, No. 7 (1999); S.A. Fulling in *The Casimir Effect Fifty Years Later*. ed. M. Bordag (World Scientific, Singapore, 1999).
15. S.A. Fulling, *J. Phys. A* **36**, 6857 (2003).
16. R. Parentani, *Class. Quantum Grav.* **10**, 1409 (1993).
17. L.H. Ford and N.F. Svaiter, *Phys. Rev. D* **58**, 065007 (1998).
18. N. Graham and K.D. Olum, *Phys. Rev. D* **67**, 085014 (2003).
19. D. Deutsch and P. Candelas, *Phys. Rev. D* **20**, 3063 (1979).
20. G. Kennedy, R. Critchley, and J.S. Dowker, *Ann. Phys. (NY)* **125**, 346 (1990).
21. P. Candelas, *Ann. Phys. (NY)* **143**, 241 (1982); *Ann. Phys. (NY)* **167**, 257 (1986).
22. R. Estrada and R.P. Kanwal, *A Distributional Approach to Asymptotics: Theory and Applications* (Birkhäuser, Boston, 2002).