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http://www.txdigital.com/math311/8-2-13.dvi
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Statement of Problem:

> Show that the Fredholm theorem (†) implies that the dimension of the space spanned by the rows of a matrix equals that of the space spanned by the columns. (This is Theorem 2 of Sec. 5.4.) HINT: Apply Theorem 1 of Sec. 5.4 (the equation relating rank to nullity) to the linear functions represented by A and A^{t}.

Begin by restating (†):

> $A\vec{x} = \vec{y}$ is solvable if and only if $\vec{y} \cdot \vec{z} = 0$ for all solutions of $A^{\mathrm{t}}\vec{z} = \vec{0}$.

To answer the problem at hand, we must show that dim range(A^{t}) = dim range(A). To this end, we employ Theorem 2 of 5.4, which states:

$$\dim \text{kernel} + \dim \text{range} = \dim \text{domain}.$$

Because $A\vec{x} = \vec{y}$, the dimension of the range of A is $\dim \vec{y}$. For $\vec{y} \cdot \vec{z} = 0$ (the second clause of the Fredholm theorem) to make sense, $\dim \vec{y} = \dim \vec{z}$. Therefore, $\dim \text{range}(A) = \dim \vec{z}$. However, $\dim \vec{z}$ is also equal to the domain of A^{t}. So $\dim \text{domain}(A^{\mathrm{t}}) = \dim \text{range}(A)$.

The second portion of the theorem states that $A^{\mathrm{t}}\vec{z} = 0$ for all $\vec{z}$. Therefore, $\dim \text{kernel} = 0$, for A^{t}.

Because $\dim \text{kernel}(A^{\mathrm{t}}) = 0$,

$$\dim \text{range}(A^{\mathrm{t}}) = \dim \text{domain}(A^{\mathrm{t}}).$$

Which means that

$$\dim \text{range}(A^{\mathrm{t}}) = \dim \text{range}(A).$$

Therefore, the Fredholm theorem implies that the dimension of the space spanned by the rows of a matrix equals that of the space spanned by the columns.