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Problem 8.1.9 **4/24/2002**

http://people.tamu.edu/~pdq981a/8.1.9.dvi
http://people.tamu.edu/~pdq981a/8.1.9.pdf

Let $M = \begin{pmatrix} 2 & 2 \\ 8 & 2 \end{pmatrix}$.

(a) Find the eigenvalues of M and the corresponding eigenvectors.
(b) Solve the differential equation

$$\frac{d\vec{x}}{dt} = M\vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution:

Part A:

$M = \begin{pmatrix} 2 & 2 \\ 8 & 2 \end{pmatrix}$

Let's find the eigenvalues of M, using the equation $\det(\vec{M} - \vec{\lambda}) = 0$:

$\det\begin{pmatrix} 2-\lambda & 2 \\ 8 & 2-\lambda \end{pmatrix} = 0$

$(2-\lambda)(2-\lambda) - 16 = 4 + \lambda^2 - 4\lambda - 16 = \lambda^2 - 4\lambda - 12 = (\lambda - 6)(\lambda + 2) = 0$

Thus, $\lambda_1 = 6$ and $\lambda_2 = -2$.

Now, let's find the eigenvectors of M, using the equation $(M - \lambda)\vec{v} = \vec{0}$.

$\lambda_1 = 6:$

$$\begin{pmatrix} 2-6 & 2 \\ 8 & 2-6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Reduce the matrix:

$$\begin{pmatrix} -4 & 2 \\ 8 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 8 & -4 \\ 8 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$$

Thus, $2v_1 - v_2 = 0$, or $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$\lambda_2 = -2:$

$$\begin{pmatrix} 2+2 & 2 \\ 8 & 2+2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Reduce the matrix:

$$\begin{pmatrix} 4 & 2 \\ 8 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 8 & 4 \\ 8 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

Thus, $2v_1 + v_2 = 0$, or $\vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Part B:

$\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$

$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$

$\vec{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ 2c_1 - 2c_2 \end{pmatrix}$

So, $c_1 = \frac{1}{4}$, $c_2 = \frac{3}{4}$

Thus, $\vec{x}(t) = \frac{1}{4} \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{6t} + \frac{3}{4} \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$.