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Problem 8.2.10 **4/24/2002**

http://people.tamu.edu/~pdq981a/8.2.10.dvi
http://people.tamu.edu/~pdq981a/8.2.10.pdf

Suppose that at a point $\vec{x}_0$ in R^2, both first-order partial derivatives of a function $f(\vec{x})$ are zero, and the matrix of second-order partial derivatives is

$$Q_{\vec{x}_0} f = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$

(a) Is $\vec{x}_0$ the location of a maximum, a minimum, or neither? Explain.

(b) Let C be a constant close to (but not equal to) $f(\vec{x}_0)$. What kind of conic section (ellipse, hyperbola, or parabola) would you expect the graph of the relation $f(\vec{x}) = C$ to resemble?

Solution:

$Q_{\vec{x}_0} f = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$

Q is the Hessian matrix of f, or more specifically $\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$, assuming $\vec{x} = (x, y)$.

Part A:

To use the second-derivative test (pp. 425-426), let's find the eigenvalues of Q:

$\det \begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix} = (2-\lambda)(2-\lambda) - 9 = 4 + \lambda^2 - 4\lambda - 9 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0$

Thus, $\lambda_1 = 5$, and $\lambda_2 = -1$.

Since the two eigenvalues are of different sign, f forms a saddle point at the point $\vec{x}_0$.

Part B:

If we convert Q to quadratic form, we have

$2x^2 + 6xy + 2y^2$

Using this and the Corollary to Theorem Q (pp. 422-423),

$A = 2, B = 6, C = 2.$

$d = -\frac{1}{4}(B^2 - 4AC) = -5.$

$t = A + C = 4.$

Since $d < 0$, f forms a hyperbola near the point $\vec{x}_0$.