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Assignment #13 8.1.12

http://people.tamu.edu/~rdw1934/8112.pdf

Let A be an n by n matrix whose eigenvalues are distinct and nonzero. Prove that A has exactly 2^n different diagonalizable square roots (that is, matrices B such that $B^2 = A$; complex entries are allowed). Do this by considering the action of B^2 on the elements of an eigenbasis in $\mathbf{C}^n$ for B. (You may ignore the possibility of nondiagonalizable square roots.)

$$\sqrt{A} = f(A) = Uf(D)U^{-1}$$

Where U is the matrix composed of the eigenvectors as columns. We know that these vectors are distinct as the corresponding eigenvalues are distinct. The matrix D is the matrix whose diagonal is the eigenvalues of A.

$$\sqrt{A} = B = U\sqrt{D}U^{-1} = U\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}^{1/2} U^{-1} = U\begin{pmatrix} \pm\sqrt{\lambda_1} & 0 & \ldots & 0 \\ 0 & \pm\sqrt{\lambda_2} & \ldots & 0 \\ \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & \pm\sqrt{\lambda_n} \end{pmatrix} U^{-1}$$

Thus, B depends on what sign we choose for the square roots of the eigenvalues.

There are n eigenvalues and the choice is binary, thus there are 2^n possibilities for the matrix $\sqrt{D}$.

For example…

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad \lambda = 1, 2 \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\sqrt{D} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \text{or} \begin{pmatrix} -1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \text{or} \begin{pmatrix} 1 & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \text{or} \begin{pmatrix} -1 & 0 \\ 0 & -\sqrt{2} \end{pmatrix}$$

After performing the change of basis calculations, we can find 4 distinct square root matrices…

$$B = \begin{pmatrix} 1 & \sqrt{2}-1 \\ 0 & \sqrt{2} \end{pmatrix} \text{or} \begin{pmatrix} -1 & \sqrt{2}+1 \\ 0 & \sqrt{2} \end{pmatrix} \text{or} \begin{pmatrix} 1 & -\sqrt{2}-1 \\ 0 & -\sqrt{2} \end{pmatrix} \text{or} \begin{pmatrix} -1 & 1-\sqrt{2} \\ 0 & -\sqrt{2} \end{pmatrix}$$

It can be shown that $B^2=A$ for these 4 matrices. With a 2 by 2 matrix, there are 4 square roots.