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Assignment 3, Problem 3.4.10
http://www.aboynamedpat.com/math/assignment3_4_10.pdf

Problem:

(a) Prove the theorem that $d_{\vec{x}_0}L = L$ if L is linear. What does this say about the relation between the matrix representing L and the partial derivatives of L?

(b) Formulate and prove the corresponding theorem about affine functions.

(a) We need to define L so let's choose to define it as $L : R^2 \to R^2$ so we can say that $L(\vec{x}) = A(\overrightarrow{x})$.

$$L(\vec{x}) = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_{11}x + L_{12}y \\ L_{21}x + L_{22}y \end{pmatrix}$$

Now we'll find dL.

$$dL(\vec{x}) = \begin{pmatrix} \frac{\partial L_1}{\partial x} & \frac{\partial L_1}{\partial y} \\ \frac{\partial L_2}{\partial x} & \frac{\partial L_2}{\partial y} \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = L$$

(b) An affine function ($N(\vec{x})$) is a linear function ($L(\vec{x})$) that does not pass through the origin because of a constant added to a linear function ($N(\vec{x}) = L(\vec{x}) + \vec{c}$). We'll do the same thing as above and when we deferentiate the constants will turn to zero.

$$N(\vec{x}) = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} L_{11}x + L_{12}y + c_1 \\ L_{21}x + L_{22}y + c_2 \end{pmatrix}$$

$$dN(\vec{x}) = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = L$$

If we do a classic affine approximation we will see that the affine approximation is the same as $N(\vec{x})$.

$$N(\vec{x}) \approx N(\vec{x}_0) + dN(\vec{x} - \vec{x}_0) = L(\vec{x}_0) + \vec{c} + L(\vec{x}) - L(\vec{x}_0) = L(\vec{x}) + \vec{c} \text{ which is actually equal to } N(\vec{x}),$$

thus showing that the affine approximation of an affine function is equivalent to the affine function.