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Assignment 3, Problem 3.2.19
http://www.aboynamedpat.com/math/assignment3_2_19.pdf

The matrix $X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ represents a 90^{deg} counterclockwise rotation around the x axis in$\mathbb{R}^3$.

(a) Find the corresponding matrices Y and Z for rotations around the other axes.

(b) Calculate XYX^{-1}. Rotate a book in your hands to convince yourself that the result is correct.

(c) Deduce geometrically the five analogous products,

$$YZY^{-1}, \quad ZXZ^{-1}, \quad YXY^{-1}, \quad ZYZ^{-1}, \quad XZX^{-1}.$$

(a) We can find the corresponding matrices by observing what our outputs should be, given out inputs. To rotate around the y axis counterclockwise, we know that our output in $\mathbb{R}^3$ will have to be $\begin{pmatrix} z \\ y \\ -x \end{pmatrix}$ from the input vector $\overrightarrow{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Knowing how vector multiplication works, we can deduce that the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ is the linear operator Y onto $\overrightarrow{v}$ which will give us the wanted output. This same method can be used to find Z.

$$Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Using row reducing methods, we can find that the inverse of X is $X^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. If we apply this operator to $\overrightarrow{v}$ we get that $X^{-1}\overrightarrow{v} = \begin{pmatrix} x \\ z \\ -y \end{pmatrix}$ which is the vector $\overrightarrow{v}$ rotated clockwise around the x axis. Multiplying three matrices together is easier by multiplying the first two together first and then multiply that product by the final matrices. Carrying out XY gives us $XY = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $XYX^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. When we apply this new linear operator to $\overrightarrow{v}$ we can find that $XYX^{-1}\overrightarrow{v} = \begin{pmatrix} -y \\ x \\ z \end{pmatrix}$ which is the equivalent of the operator Z. Using a book for a visual example (using the 3 sides as 3 different planes), we can apply these geometric transformations one by one and compare our result to our calculated transformation. We find that this is correct.

(c) Since we're rotating the axis three times, twice along the same axis just in different directions, we can deduce geometrically the new operations. In every case, the axis that is never rotated around will remain the same in the end because the axis will always "fall" back into place. Also, the axis that are rotated about twice (once counterclockwise, once clockwise) are now given the value of the axis that was rotated once, only negative. The axis that was rotated once receives the value of the axis that was rotated twice.

$$YZY^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad ZXZ^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad YXY^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$ZYZ^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad XZX^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Applying these operators to $\overrightarrow{v}$ shows a more visual interpretation:

$$YZY^{-1}\overrightarrow{v} = \begin{pmatrix} x \\ -z \\ y \end{pmatrix}, \quad ZXZ^{-1}\overrightarrow{v} = \begin{pmatrix} z \\ y \\ -x \end{pmatrix}, \quad YXY^{-1}\overrightarrow{v} = \begin{pmatrix} y \\ -x \\ z \end{pmatrix},$$

$$ZYZ^{-1}\overrightarrow{v} = \begin{pmatrix} x \\ z \\ -y \end{pmatrix}, \quad XZX^{-1}\overrightarrow{v} = \begin{pmatrix} -z \\ y \\ x \end{pmatrix}.$$