Patrick Barrett (p-barrett@calclab.math.tamu.edu)
Assignment 4, Problem 4.1.8
http://www.aboynamedpat.com/math/assignment4_1_8.pdf

Problem:

Prove that the functions $\{e^t, e^{2t}, e^{3t}\}$are independent (as elements of $\mathcal{C}^2(-\infty,\infty)$, or of $\mathcal{C}^2(a,b)$ over any interval $(a,b) \subseteq \mathbf{R}$). Possible alternative strategies:

(A) Set and $z = e^t$ use the corresponding fact for polynomials.

(B) Prove it directly: Evaluate e^{kt} for three different values of t and the three relevant values of k , getting a matrix 3×3 matrix. Show that if the three functions are linerly independent, then the determinant of this matrix must be zero. But then use properties of the exponential function to show that the determinant is not zero.

(A) To prove that the functions are independent, we need to set them up as a linear combination equal to zero.

$$r_1 e^t + r_2 e^{2t} + r_3 e^{3t} = 0$$

Now, to make this simplier, we set $z = e^t$ and substitute in. Since we know that the funtions are independent only when $r_1 = 0$, $r_2 = 0$, $\ldots$, $r_k = 0$, we are looking to see if this substitution will visually make this case obvious.

$$r_1 z + r_2 z^2 + r_3 z^3 = 0$$

The only way for this expression to be true is when $r_1 = 0$, $r_2 = 0$, $r_3 = 0$. Otherwise, there would be a z term leftover.

(B) (Prove directly) When we evaluate e^{kt} for 3 different values of t (here we'll choose 1, 2, and 3) and form a set of vectors with the fact that $k = 1$, 2, and 3, we get this 3×3 matrix:

$$\begin{pmatrix} e & e^2 & e^3 \\ e^2 & e^4 & e^6 \\ e^3 & e^6 & e^9 \end{pmatrix} = \mathcal{S}$$

From A, we know that these are independent functions. If we were to take the determinant of this matrix $\mathcal{S}$, we will find that the the solution will not equal zero, satisfying the problem.

$$\begin{vmatrix} e & e^2 & e^3 \\ e^2 & e^4 & e^6 \\ e^3 & e^6 & e^9 \end{vmatrix} = e^{13} + e^{11} + e^{11} - e^{13} - e^{13} - e^{10} = 2e^{11} - e^{10} - e^{13} \neq 0$$