Patrick Barrett (p-barrett@calclab.math.tamu.edu)
Problem 5.1.7
http://www.aboynamedpat.com/math/assignment5_1_7.pdf

Problem:

Prove that the $n \times n$ antisymmetric matrices form a subspace in the space of all square matrices of the same size. Find a basis for this space and the dimension of it.

First, we'll define two $n \times n$ antisymmetric matrices as

$$A = \begin{pmatrix} 0 & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ a_{12} & 0 & -a_{23} & \cdots & -a_{2n} \\ a_{13} & a_{23} & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -a_{(n-1)n} \\ a_{1n} & a_{2n} & \cdots & a_{(n-1)} & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & -b_{12} & -b_{13} & \cdots & -b_{1n} \\ b_{12} & 0 & -b_{23} & \cdots & -b_{2n} \\ b_{13} & b_{23} & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -b_{(n-1)n} \\ b_{1n} & b_{2n} & \cdots & b_{(n-1)n} & 0 \end{pmatrix}$$

To check that A and B are each a subspace of all square matrices of $n \times n$, we need to show that they are closed under the vector operations of addition and multiplication.

$$rA + B = \begin{pmatrix} 0 & -ra_{12} - b_{12} & -ra_{13} - b_{13} & \cdots & -ra_{1n} - b_{1n} \\ ra_{12} + b_{12} & 0 & -ra_{23} - b_{23} & \cdots & -ra_{2n} - b_{2n} \\ ra_{13} + b_{13} & ra_{23} + b_{23} & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -ra_{(n-1)n} - b_{(n-1)n} \\ ra_{1n} + b_{1n} & ra_{2n} + b_{2n} & \cdots & ra_{(n-1)n} + b_{(n-1)n} & 0 \end{pmatrix}$$

If we state that $C = rA + B$ then $c = ra + b$ and the matrix is closed. Thus showing that the antisymetric matrices of dimension $n \times n$ is a subspace of the sqare matrix $n \times n$.

$$C = \begin{pmatrix} 0 & -c_{12} & -c_{13} & \cdots & -c_{1n} \\ c_{12} & 0 & -c_{23} & \cdots & -c_{2n} \\ c_{13} & c_{23} & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -c_{(n-1)n} \\ c_{1n} & c_{2n} & \cdots & c_{(n-1)} & 0 \end{pmatrix}$$

The basis of this space is as follows:

$$\left\{ \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ldots \text{etc.} \right\}$$

and the dimension of this basis will be $\frac{n(n-1)}{2}$. The divide by 2 is due to the fact that one half of your matrix is the negative of the other half and anything on one side will be complimented by the same value on the other.