2.2.25

Prove that if A is anisymmetric, then its diagonal elements, A_{11}, A_{22} …, are all 0.

An *antisymmetric* or *skew-symmetric* matrix, A, has the property such that $A^t = -A$. Where A^t represents the transpose of matrix A, and $-A$ has all elements opposite in sign those of A. The transpose operation switches the rows and columns of a matrix, that is $(A^t)_{jk} = A_{kj}$, where j and k represent the row and column indices.

For Example:

$$A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \qquad A^t = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \qquad -A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

Since $A^t = -A$, matrix A is *antisymmetric*.

In a more general case:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \qquad A^t = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

$$-A = \begin{pmatrix} -A_{11} & -A_{12} & \cdots & -A_{1n} \\ -A_{21} & -A_{22} & \cdots & -A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{n1} & -A_{n2} & \cdots & -A_{nn} \end{pmatrix}$$

Equating A^t and $-A$ (by equating elements):

$$A^t = -A$$
$$A^t + A = 0$$

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 2A_{11} & A_{21}+A_{12} & \cdots & A_{n1}+A_{1n} \\ A_{12}+A_{21} & 2A_{22} & \cdots & A_{n2}+A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n}+A_{n1} & A_{2n}+A_{n2} & \cdots & 2A_{nn} \end{pmatrix}$$

It follows, that for each element A_{jk}, where $j \neq k$, $A_{jk} + A_{kj} = 0$ is satisfied when $A_{jk} = -A_{kj}$. However when $j = k$, as along the diagonal, $2A_{jk} = 2A_{kj} = 2A_{nn} = 0$. This is only satisfied when the elements are 0. Thus, for all antisymmetric matrices, the diagonal elements are identically zero.