2.5.11

Problem:

Is it possible to define a vector division operation inverse to the cross product, so that

$$\frac{\vec{u}\times\vec{v}}{\vec{v}}=\vec{u}$$

for all $\vec{u}$ in $\mathbf{R}^3$?

Solution:

No.

The cross product of two vectors is a vector perpendicular to both, in $\mathbf{R}^3$ the cross product is defined as:

$$\vec{u}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ u_x & u_y & u_z\\ v_x & v_y & v_z\end{vmatrix}=(u_y v_z-u_z v_y)\hat{i}+(u_z v_x-u_x v_z)\hat{j}+(u_x v_y-u_y v_x)\hat{k}$$

This vector can be written as a column vector.

$$\begin{pmatrix}u_y v_z-u_z v_y\\ u_z v_x-u_x v_z\\ u_x v_y-u_y v_x\end{pmatrix}$$

This column vector can be expressed as the product of a two matrices made of the components of the original vectors $\vec{u}$ and $\vec{v}$.

$$\vec{u}\times\vec{v}=\begin{pmatrix}0 & -u_z & u_y\\ u_z & 0 & -u_x\\ -u_y & u_x & 0\end{pmatrix}\begin{pmatrix}v_x\\ v_y\\ v_z\end{pmatrix}=\begin{pmatrix}0 & v_z & -v_y\\ -v_z & 0 & v_x\\ v_y & -v_x & 0\end{pmatrix}\begin{pmatrix}u_x\\ u_y\\ u_z\end{pmatrix}=\begin{pmatrix}u_y v_z-u_z v_y\\ u_z v_x-u_x v_z\\ u_x v_y-u_y v_x\end{pmatrix}$$

This is obtained by letting one of the two original vectors be a column vector which is operated on by a 3x3 matrix made from the components of the other vector.

$$\vec{u}\times\vec{v}=\begin{pmatrix}0 & v_z & -v_y\\ -v_z & 0 & v_x\\ v_y & -v_x & 0\end{pmatrix}\begin{pmatrix}u_x\\ u_y\\ u_z\end{pmatrix}=\begin{pmatrix}u_y v_z-u_z v_y\\ u_z v_x-u_x v_z\\ u_x v_y-u_y v_x\end{pmatrix}$$

So attempting to solve for $\vec{u}$ as a function of $\vec{v}$ and $\vec{u}\times\vec{v}$ yields:

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} 0 & v_z & -v_y \\ -v_z & 0 & v_x \\ v_y & -v_x & 0 \end{pmatrix}^{-1} \begin{pmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{pmatrix}$$

To see whether or not the $\vec{v}$ matrix can be inverted, its determinant is taken:

$$\begin{vmatrix} 0 & v_z & -v_y \\ -v_z & 0 & v_x \\ v_y & -v_x & 0 \end{vmatrix} = 0 + v_z v_x v_y + (-v_y)(-v_z)(-v_x) - (-v_y)(0)(v_y) - (v_z)(-v_z)(0) - (0)(v_x)(-v_x) = 0$$

However, since the determinant of this matrix is identically zero, the matrix is singular and cannot be inverted. Thus, $\vec{u}$ cannot be found.