Problem:

Let $x = u^2 - \cos v$, $y = e^u + v^2$, $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \sin t \\ 2\cos t \end{pmatrix}$. Calculate $\frac{dx}{dt}$ and $\frac{dy}{dt}$ when $t = \pi$.

Solution:

Let $f = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix} = \begin{pmatrix} u^2 - \cos v \\ e^u + v^2 \end{pmatrix}$

Compute the Jacobian of f, then substitute the values of $u(t)$ and $v(t)$:

$$J(f) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 2u & \sin v \\ e^u & 2v \end{pmatrix}$$

$$J(f)\Big|_{\substack{u=\sin t \\ v=2\cos t}} = \begin{pmatrix} 2\sin t & \sin(2\cos t) \\ e^{\sin t} & 4\cos t \end{pmatrix}$$

Compute the derivatives of $u(t)$ and $v(t)$ with respect to t:

$$\frac{d}{dt}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \cos t \\ -2\sin t \end{pmatrix}$$

By the Chain Rule:

$$J(f)\Big|_{\substack{u=\sin t \\ v=2\cos t}} \frac{d}{dt}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$$

$$\begin{pmatrix} 2\sin t & \sin(2\cos t) \\ e^{\sin t} & 4\cos t \end{pmatrix}\begin{pmatrix} \cos t \\ -2\sin t \end{pmatrix} = \begin{pmatrix} 2\sin t\cos t - 2\sin t\sin(2\cos t) \\ e^{\sin t}\cos t - 8\sin t\cos t \end{pmatrix}$$

Substituting $t = \pi$:

$$\begin{pmatrix} 2\sin\pi\cos\pi + 2\sin\pi\sin(2\cos\pi) \\ e^{\sin\pi}\cos\pi - 8\sin\pi\cos\pi \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\frac{dx}{dt}\Big|_{t=\pi} = 0 \qquad \frac{dy}{dt}\Big|_{t=\pi} = -1$$