Problem:

Define the linear operator $L : P_4 \to P_4$ by

$$Lp(t) = 3p(t) - p'(t).$$

(As usual, p' is the derivative of the polynomial p.) Find the matrix that represents L with respect to the standard basis $\{t^4, t^3, t^2, t, 1\}$.

Solution:

Calculate:

$$Lp(t^4) = 3t^4 - 4t^3$$
$$Lp(t^3) = 3t^3 - 3t^2$$
$$Lp(t^2) = 3t^2 - 2t$$
$$Lp(t) = 3t - 1$$
$$Lp(1) = 3$$

Put these results into a matrix, where each row represents a power of t and each column represents the linear operator L performed on a power of t. The values down a column are the multiples of each power of t that result from the linear operation.

$$\begin{array}{c} \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{array}\begin{array}{c} \begin{array}{ccccc} L(t^4) & L(t^3) & L(t^2) & L(t) & L(1) \end{array} \\ \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ -4 & 3 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix} \end{array}$$

Verification:

$$Lp(6t^4 + 9) = 3(6t^4 + 9) - 24t^3 = 18t^4 - 24t^3 + 27$$

By Matrix

$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ -4 & 3 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 9 \end{pmatrix} = \begin{pmatrix} 18 \\ -24 \\ 0 \\ 0 \\ 27 \end{pmatrix}$$