5.3.4
Problem:

Consider *Poisson's equation*, $\nabla^2\phi = \rho$. (Physically, $\rho(\vec{r})$ is a given charge distribution and $\phi(\vec{r})$ is the electrical potential to be found.)

(a) Note that if ϕ is a solution, then $\phi + e^y \sin x$ is also a solution (with the same ρ). Explain what this has to do with the terms "injective" and "kernel".

(b) Suppose that the equation holds inside a sphere with center at the origin, and that the radial derivative $\frac{\partial\phi}{\partial r} = \hat{n}\cdot\nabla\phi$ (the directional derivative perpendicular to the sphere) equals 0 everywhere on the sphere. (This means on the surface, not the whole interior!) In Exercise 7.5.6 we shall show that the integral of ρ over the interior of the sphere must then equal 0. (Physically, the net charge is zero.) Explain what this phenomenon has to do with the terms "range" and "onto". HINT: Think of ∇^2 as a linear operator on a domain consisting of functions whose normal derivatives vanish on the sphere; see also the next exercise.

Solution:

(a) ∇^2 stands for the *Laplacian operator*, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

A function is *injective* or *one-to-one* if $L(\vec{x}) = L(\vec{y})$ implies $\vec{x} = \vec{y}$. The solution must also be unique.

$\rho = \nabla^2\phi$

And since both solutions have the same ρ:

$\rho = \nabla^2(\phi + e^y \sin x)$

$\rho = \nabla^2\phi + \nabla^2(e^y \sin x)$

$\rho = \rho + \nabla^2(e^y \sin x)$

$\nabla^2(e^y \sin x) = 0$

Since two different inputs to the *Laplacian operator* both map to the same output, ρ, the operator is not injective (one-to-one) and the kernel is not $\vec{0}$.

(b) If ∇^2 is a linear operator, L, which maps functions on domain D into codomain W, then the function is *onto* or *surjective* if the range of $L: D \to W$ is *all* of W. Here, the domain D consists of functions $\phi(\vec{r})$ whose normal derivatives vanish on the sphere (electric potential would not change if one were to move radially inward or outward from the surface of the sphere). The codomain W consists of all values of $\rho(\vec{r})$. However, the integral of ρ over the interior of the sphere must equal 0 (since the net charge is 0), and the range consists only of values of $\rho(\vec{r})$ which satisfy this. Therefore the range is not all of W and the function is not onto.