Problem:

Define coordinates (u,v) in a suitable region of the (x,y) plane by

$$x = u^2 v \qquad y = u + v^2 .$$

(a) To be "suitable", the region must stay away from the places where the determinant of the Jacobian of the coordinate transformation equals 0. Find these dangerous places. (The final answer should be one or more equations relating x and y, with u and v eliminated.)

(b) Calculate the partial derivatives of u and v with respect to x and y. (The formulas for the answers will contain u and v.

(c) In the (x,y) plane, at the point $(4,3)$ (corresponding to $u=2$, $v=1$), draw the tangent vectors to the coordinate curves and the normal vectors to the coordinate "surfaces" (which are curves in dimension 2). (Don't normalize these vectors to unit length.) Then plot the curve $u=2$, the curve $v=1$, and (with dashed lines) the dangerous curves you found in (a). Comment on the relation among these curves.

Solution:

(a) $\dfrac{\partial x}{\partial u} = 2uv \qquad \dfrac{\partial x}{\partial v} = u^2 \qquad \dfrac{\partial y}{\partial u} = 1 \qquad \dfrac{\partial y}{\partial v} = 2v$

$$J = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 2uv & u^2 \\ 1 & 2v \end{pmatrix} \qquad \begin{vmatrix} 2uv & u^2 \\ 1 & 2v \end{vmatrix} = 4uv^2 - u^2 = 0$$

$$4uv^2 = u^2$$
$$4v^2 = u$$
$$4(y-u) = u$$
$$4y - 4u = u$$
$$4y = 5u$$

$$4\left(\frac{x}{u^2}\right)^2 = u$$
$$4x^2 = u^5$$
$$(4x^2)^{\frac{1}{5}} = u$$
$$4y = 5(4x^2)^{\frac{1}{5}}$$

Dangerous curves: $y = \dfrac{5(2x)^{\frac{2}{5}}}{4}$

(b)

$$J=\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix} \qquad J^{-1}=\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix} \qquad \begin{pmatrix}2uv & u^2\\ 1 & 2v\end{pmatrix}^{-1}=\begin{pmatrix}\frac{-2v}{u(u-4v^2)} & \frac{u}{u-4v^2}\\ \frac{1}{u(u-4v^2)} & \frac{-2v}{u-4v^2}\end{pmatrix}$$

$$\frac{\partial u}{\partial x}=\frac{-2v}{u(u-4v^2)} \qquad \frac{\partial u}{\partial y}=\frac{u}{u-4v^2} \qquad \frac{\partial v}{\partial x}=\frac{1}{u(u-4v^2)} \qquad \frac{\partial v}{\partial y}=\frac{-2v}{u-4v^2}$$

(c)

Tangent Vectors:

$$\begin{pmatrix}\frac{\partial x}{\partial u}\\ \frac{\partial y}{\partial u}\end{pmatrix}=\begin{pmatrix}2uv\\ 1\end{pmatrix}\Bigg|_{(u,v)=(2,1)}=\begin{pmatrix}4\\ 1\end{pmatrix}$$

$$\begin{pmatrix}\frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial v}\end{pmatrix}=\begin{pmatrix}u^2\\ 2v\end{pmatrix}\Bigg|_{(u,v)=(2,1)}=\begin{pmatrix}4\\ 2\end{pmatrix}$$

Normal Vectors:

$$\begin{pmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{-2v}{u(u-4v^2)}\\ \frac{u}{u-4v^2}\end{pmatrix}\Bigg|_{(u,v)=(2,1)}=\begin{pmatrix}\frac{1}{2}\\ -1\end{pmatrix}$$

$$\begin{pmatrix}\frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{1}{u(u-4v^2)}\\ \frac{-2v}{u-4v^2}\end{pmatrix}\Bigg|_{(u,v)=(2,1)}=\begin{pmatrix}-\frac{1}{4}\\ 1\end{pmatrix}$$

Curves

$u=2$

$x=4v$

$y=2+v^2$

$y=2+\frac{x^2}{16}$

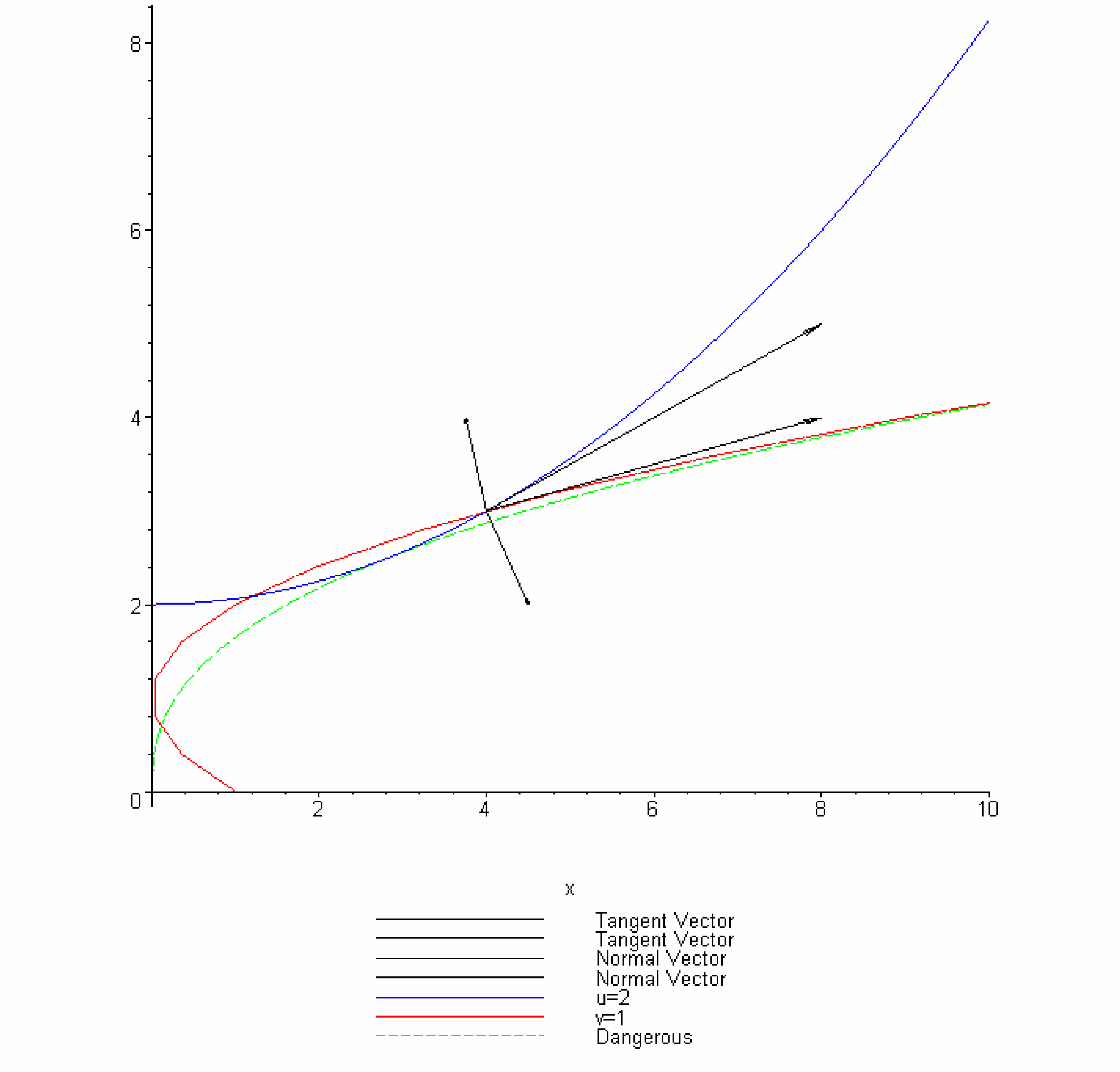
$v=1$

$x=u^2(1)$

$y=u+(1)^2$

$x=(y-1)^2$

$$y=\frac{5(2x)^{\frac{2}{5}}}{4}$$



The two curves $u=2$ and $v=1$ intersect at the point $(4,3)$ from which the two tangent and two normal vectors originate. The dangerous curve is below the curve $v=1$ and appears to converge to the curve $v=1$. (This is best viewed online in color).