Problem:
Find the area between the curves $u = 1$ and $u = 2$ in the elliptical coordinate system of Exercise 4.2.1. Sketch those curves and two curves of constant v.

Exercise 4.2.1:

The formulas

$$x = \cosh u \cos v$$
$$y = \sinh u \sin v$$

define *elliptic coordinate* (u, v) in the $x - y$ plane, with the ranges $u \geq 0$, $0 \leq v \leq 2\pi$.

Solution:
First find $\dfrac{\partial(x, y)}{\partial(u, v)}$:

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \sinh u \cos v & -\cosh u \sin v \\ \cosh u \sin v & \sinh u \cos v \end{vmatrix} = \sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v$$

By applying trigonometric identities, this can be simplified:

$$\sin^2 v + \cos^2 v = 1$$
$$\cosh^2 u - \sinh^2 u = 1$$
$$\sinh^2 u(1 - \sin^2 v) + (1 + \sinh^2 u)\sin^2 v$$
$$\sinh^2 u - \sinh^2 u \sin^2 v + \sin^2 v + \sinh^2 u \sin^2 v$$
$$\sinh^2 u + \sin^2 v$$

The area can be found in the following way:

$$A = \int_0^{2\pi} \int_1^2 \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$
$$A = \int_0^{2\pi} \int_1^2 (\sinh^2 u + \sin^2 v) du dv$$

Next apply the following trigonometric identities to aid in integration:

$$\sinh^2 u = \frac{\cosh 2u - 1}{2}$$
$$\sin^2 v = \frac{1 - \cos 2v}{2}$$

Yielding this:

$$A = \frac{1}{2}\int_0^{2\pi}\int_1^2 \cosh 2u - 1 + 1 - \cos 2v du dv$$

Now evaluate the integral:

$$A = \frac{1}{2}\int_0^{2\pi}\int_1^2 \cosh 2u - \cos 2v du dv$$

$$A = \frac{1}{2}\int_0^{2\pi}\left[\frac{\sinh 2u}{2} - u\cos 2v\right]_1^2 dv$$

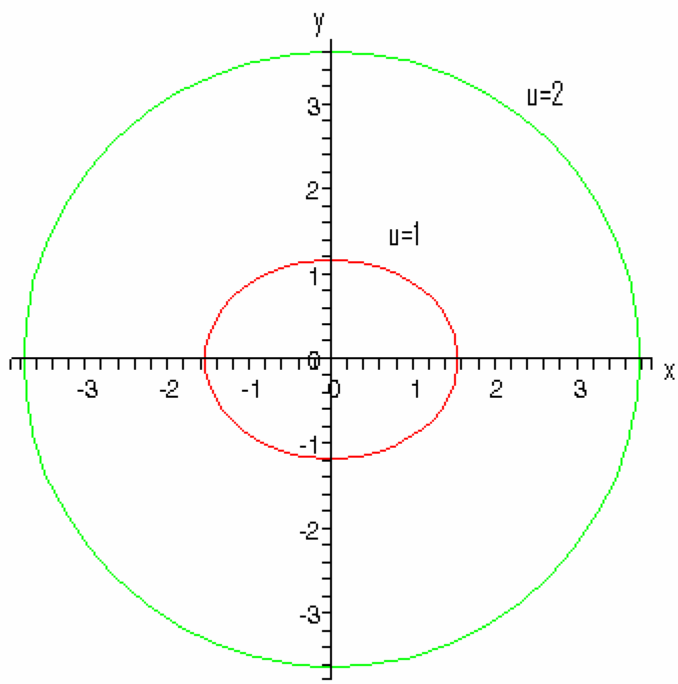
$$A = \frac{1}{2}\int_0^{2\pi}\frac{\sinh 4}{2} - 2\cos 2v - \frac{\sinh 2}{2} + \cos 2v dv$$

$$A = \frac{1}{2}\left[v\left(\frac{\sinh 4}{2} - \frac{\sinh 2}{2}\right) - \sin 2v + \frac{\sin 2v}{2}\right]_0^{2\pi}$$

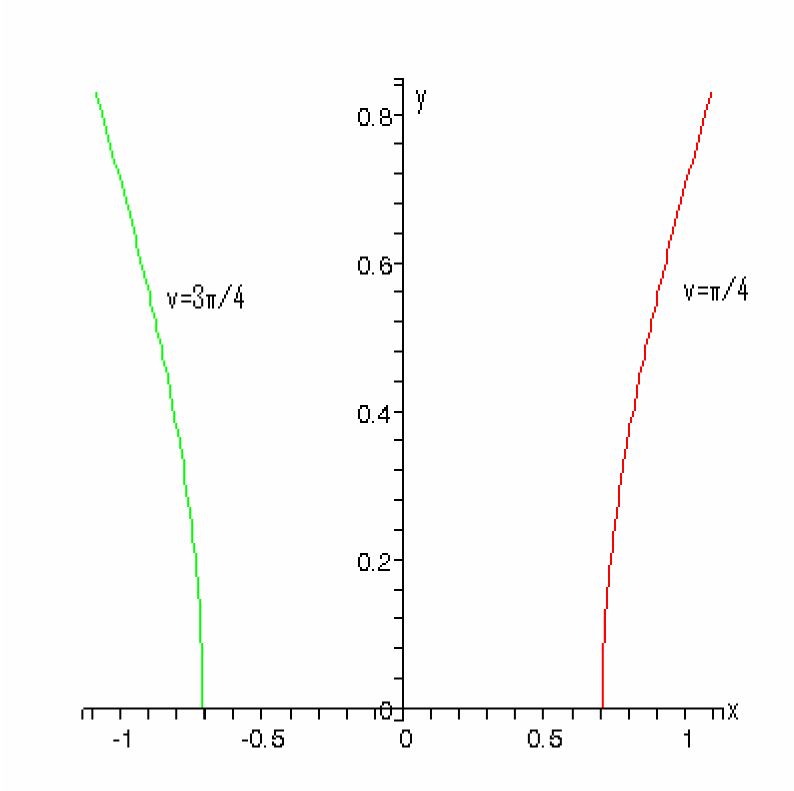
$$A = \frac{1}{2}\left[2\pi\left(\frac{\sinh 4}{2} - \frac{\sinh 2}{2}\right) - \sin(4\pi) + \frac{\sin(4\pi)}{2} + \sin(0) - \frac{\sin(0)}{2}\right]$$

$$A = \frac{\pi}{2}(\sinh 4 - \sinh 2)$$

$$A \approx 37.17 units^2$$



The inner curve is $u = 1$ and the outer curve is $u = 2$. The area found is that which is between them.



The curve on the right is $v = \dfrac{\pi}{4}$ and the curve on the left is $v = \dfrac{3\pi}{4}$, where $0 \le u \le 1$.