Problem:

Assume that A is a diagonalizable matrix (independent of t) and that all its eigenvalues are positive. Show that the solution of the second-order homogeneous linear differential equation

$$\frac{d^2\vec{x}}{dt^2} = -A\vec{x}; \qquad \vec{x}(0) = \vec{x}_0, \qquad \vec{x}'(0) \equiv \left.\frac{d\vec{x}}{dt}\right|_{t=0} = \vec{v}_0,$$

can be written in terms of trigonometric functions of the matrix $t\sqrt{A}$ acting on the vectors $\vec{x}_0$ and $\vec{v}_0$.

Solution:

One can compare this system of differential equations with that of the simple harmonic oscillator:

$$\frac{d^2x}{dt^2} + Ax = 0$$

Which has a characteristic equation:

$r^2 + A = 0$

with the following roots,

$r^2 = -A$

$r = \pm\sqrt{-A}$

$r = \pm i\sqrt{A}$

and solutions of the form:

$x = c_1 \sin(t\sqrt{A}) + c_2 \cos(t\sqrt{A})$

Begin with the diagonalized form of A,

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

which is composed of its (positive) eigenvalues λ.

Let U be the matrix composed of the eigenvectors placed side by side:

$$U = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_n \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

This matrix maps coordinates from the eigenbasis to the natural basis, so U^{-1} maps coordinates from the natural basis to the eigenbasis.

This problem can be solved by working in the eigenbasis:

$$U^{-1}\frac{d^2\vec{x}}{dt^2} = -U^{-1}A\vec{x}$$

The equation:

$$D = U^{-1}AU$$

can be solved for $U^{-1}A$, yielding

$$U^{-1}A = DU^{-1}$$

So the differential equation becomes

$$U^{-1}\frac{d^2\vec{x}}{dt^2} = -DU^{-1}\vec{x}$$

Now this can be solved in the eigenbasis:

Allow the substitution $\vec{v} = U^{-1}\vec{x}$:

$$\frac{d^2\vec{v}}{dt^2} = -D\vec{v}$$

$$\frac{d^2\vec{v}}{dt^2} = -\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}\vec{v}$$

$$\frac{d^2\vec{v}}{dt^2} = \begin{pmatrix} -\lambda_1 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & -\lambda_n \end{pmatrix}\vec{v}$$

$$\frac{d^2}{dt^2}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} -\lambda_1 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & -\lambda_n \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Which yields many differential equations of the same form:

$$\frac{d^2 v_n}{dt^2} + \lambda_n v_n = 0$$

With characteristic equations:

$$r^2 + \lambda_n = 0$$
$$r^2 = -\lambda_n$$
$$r = \pm i\sqrt{\lambda_n}$$

We have assumed in the problem statement that all values of λ are positive.

So solutions to each of these equations will be of the form:

$$v_n = c_1 \sin(t\sqrt{\lambda_n}) + c_2 \cos(t\sqrt{\lambda_n})$$
$$\vec{v} = c_1 \sin(t\sqrt{D}) + c_2 \cos(t\sqrt{D})$$

Converting back to the natural basis

$$U^{-1}\vec{x} = c_1 \sin(t\sqrt{D}) + c_2 \cos(t\sqrt{D})$$

And applying $U \sin(t\sqrt{D})U^{-1} = \sin(t\sqrt{A})$

$$\vec{x} = U(U^{-1}c_1 \sin(t\sqrt{A})U + U^{-1}c_2 \cos(t\sqrt{A})U)$$
$$\vec{x} = c_1 \sin(t\sqrt{A}) + c_2 \cos(t\sqrt{A})$$
$$\vec{x}' = c_1\sqrt{A}\cos(t\sqrt{A}) - c_2\sqrt{A}\sin(t\sqrt{A})$$

Where initial conditions $\vec{x}_0$ and $\vec{v}_0$ are needed to fix a solution.

$$\vec{x}(0) = \vec{x}_0 = c_1 \sin((0)\sqrt{A}) + c_2 \cos((0)\sqrt{A})$$
$$\vec{x}_0 = c_2$$

$$\vec{x}'(0) = \vec{v}_0 = c_1\sqrt{A}\cos((0)\sqrt{A}) - c_2\sqrt{A}\sin((0)\sqrt{A})$$
$$\vec{v}_0 = c_1\sqrt{A}$$
$$c_1 = \frac{\vec{v}_0}{\sqrt{A}}$$

Yielding a solution:

$$\vec{x} = \frac{\vec{v}_0}{\sqrt{A}}\sin(t\sqrt{A}) + \vec{x}_0 \cos(t\sqrt{A})$$