Problem:

The only eigenvalues of the matrix $B = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ are $\lambda_1 = 0$ and $\lambda_2 = -3$. Construct an orthonormal basis for $\mathbf{R}^3$ consisting of the eigenvectors of B.

Solution:

First find the eigenvectors from the eigenvalues:

$$\lambda_1 = 0$$

$$B - \lambda_1 I = \begin{pmatrix} -2-0 & 1 & 1 \\ 1 & -2-0 & 1 \\ 1 & 1 & -2-0 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$rref \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v_1 - v_3 = 0$$

$$v_1 = v_3$$

$$v_2 - v_3 = 0$$

$$v_2 = v_3$$

Yields the eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$B - \lambda_2 I = \begin{pmatrix} -2+3 & 1 & 1 \\ 1 & -2+3 & 1 \\ 1 & 1 & -2+3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$rref\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$u_1 + u_2 + u_3 = 0$$

Yields the eigenvectors $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

So, we use the Gram Schmidt procedure to construct an orthonormal basis:

Arbitrarily start by normalizing the 1st eigenvector:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \hat{u}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Find the component of the 2nd eigenvector perpendicular to $\hat{u}_1$, normalize:

$$\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{v}_2 = v_\perp + v_\parallel$$

$$v_\parallel = (\vec{v}_2 \cdot \hat{u}_1)\hat{u}_1$$

$$v_\parallel = 0$$

$$\vec{v} = v_\perp$$

$$\hat{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Find the component of the 3rd eigenvector perpendicular to $\hat{u}_1$ and $\hat{u}_2$, normalize:

$$\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{v}_3 = v_\perp + v_\parallel$$

$$\vec{v}_\parallel = (\vec{v}_3 \cdot \hat{u}_1)\hat{u}_1 + (\vec{v}_3 \cdot \hat{u}_2)\hat{u}_2$$

$$\vec{v}_\parallel = 0\hat{u}_1 + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{v}_\parallel = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$v_\perp = \vec{v}_3 - \vec{v}_\parallel$$

$$v_\perp = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$v_\perp = \frac{1}{2}\begin{pmatrix}-1\\-1\\2\end{pmatrix} \qquad \hat{u}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\-1\\2\end{pmatrix}$$

Thus, an orthonormal basis for $\mathbf{R}^3$ consists of the following vectors:

$$\hat{u}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \qquad \hat{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix} \qquad \hat{u}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\-1\\2\end{pmatrix}$$