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Assignment 12, Problem 7.5.5
http://calclab.tamu.edu/˜j-macfarlane/math311/7_5_5.pdf

Compute the integral of $\bigtriangledown \times \vec{F}$, where $\vec{F}(\vec{r}) = (x^2 z e^{2y}, yx^3, z)$, over the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$.
(a) Use Stoke's theorem to reduce the problem to an example in Sec. 6.3
(b) Use Gauss's theorem to replace the hemisphere by the disk at it's base.

Solution

(a) Stoke's theorem states that

$$\int_S (\bigtriangledown \times \vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

so if S is the hemisphere indicated then C would be a unit circle in the xy-plane.

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C x^2 z e^{2x}\, dx + yx^3\, dy + z\, dz$$

Since $z = 0$ for the whole curve, this becomes

$$\oint_C yx^3\, dy.$$

If we parametrize this curve by using cylindrical coordinates, we get

$$\int_0^{2\pi} sin\theta cos^4\theta\, d\theta = \frac{1}{5} cos^5\theta \Big|_{2\pi}^{0} = 0$$

(b) To use Gauss's theorem we need a closed surface so we take the integral over the hemisphere and the disk that closes it. We can then use this to apply Gauss's theorem. Because $\bigtriangledown \cdot (\bigtriangledown \times \vec{F}) = 0$,

$$\int_{S_{hemisphere}} \bigtriangledown \times \vec{F} \cdot d\vec{S} + \int_{S_{disk}} \bigtriangledown \times \vec{F} \cdot d\vec{S} = \int_V \bigtriangledown \cdot (\bigtriangledown \times \vec{F}) \cdot dV = 0$$

and

$$\int_{S_{hemisphere}} \bigtriangledown \times \vec{F} \cdot d\vec{S} = -\int_{S_{disk}} \bigtriangledown \times \vec{F} \cdot d\vec{S}.$$

For the disk to create a closed surface the normal vector must point down so $-\int_{S_{disk}} \bigtriangledown \times \vec{F} \cdot d\vec{S}$ with the normal down is the same as $\int_{S_{disk}} \bigtriangledown \times \vec{F} \cdot d\vec{S}$ with the normal up.

$$\int_{S_{disk}} \bigtriangledown \times \vec{F} \cdot d\vec{S} = \int_{S_{disk}} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) d\vec{S}$$

By Green's theorem, this is equal to $\oint_C \vec{F} \cdot d\vec{r}$ where C is the unit circle that bounds the disk. This is the same integral obtained by the other method.