James Macfarlane (j-macfarlane)
Assignment 13, Problem 8.1.24
http://calclab.tamu.edu/~j-macfarlane/math311/8_1_24.pdf

Find the eigenvalues and eigenvectors of the matrix. Remark upon any case where an eigenbasis of real eigenvectors does not exist.

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ \alpha & 0 & \alpha \end{pmatrix}$$

Solution

First, to find the eigenvalues, we take $det(M - \lambda) = 0$.

$$\begin{vmatrix} \alpha-\lambda & 0 & 0 \\ 0 & \alpha-\lambda & 0 \\ \alpha & 0 & \alpha-\lambda \end{vmatrix} = (\alpha-\lambda)^3 = 0$$

$$\lambda = \alpha$$

To find the eigenvectors, we take $(M - \lambda)\,\vec{v} = (\lambda - \lambda)\,\vec{v} = \vec{0}$.

$$\begin{pmatrix} \alpha-\lambda & 0 & 0 \\ 0 & \alpha-\lambda & 0 \\ \alpha & 0 & \alpha-\lambda \end{pmatrix} \vec{v} = \vec{0}$$

This reduces to

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{v} = \vec{0}.$$

Since $\alpha \neq 0$,

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

This is not a complete eigenbasis because it does not span $\mathbb{R}^3$. (If $\alpha = 0$, there would be another vector, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, which would complete the basis.)