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Assignment 13, Problem 8.2.2
http://calclab.tamu.edu/˜j-macfarlane/math311/8_2_2.pdf

Let $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$. Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

Solution

First, to find the eigenvalues, we take $det(A - \lambda) = 0$.

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ 2 & 3-\lambda & 2 \\ 2 & 2 & 3-\lambda \end{vmatrix} = (7-\lambda)(1-\lambda)^2 = 0$$

$$\lambda_1 = 7 \qquad \lambda_2 = 1$$

To find the eigenvectors, we take $(A - \lambda)\vec{v} = (\lambda - \lambda)\vec{v} = \vec{0}$. First, we use $\lambda_1 = 7$.

$$\begin{pmatrix} 3-\lambda_1 & 2 & 2 \\ 2 & 3-\lambda_1 & 2 \\ 2 & 2 & 3-\lambda_1 \end{pmatrix} \vec{v} = \begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \vec{v} = \vec{0}$$

This reduces to

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \vec{v} = \vec{0}.$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Next, we use $\lambda_2 = 1$.

$$\begin{pmatrix} 3-\lambda_2 & 2 & 2 \\ 2 & 3-\lambda_2 & 2 \\ 2 & 2 & 3-\lambda_2 \end{pmatrix} \vec{v} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \vec{v} = \vec{0}$$

This reduces to

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{v} = \vec{0}.$$

$$\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\hat{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{1^2+1^2+1^2}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{v}_{2\parallel} = (\hat{u}_1 \cdot \vec{v}_2)\,\hat{u}_1 = \frac{1}{\sqrt{3}}\,(-1+1+0)\,\hat{u}_1 = \vec{0}$$

$$\vec{v}_{2\perp} = \vec{v}_2$$

$$\hat{u}_2 = \frac{\vec{v}_{2\perp}}{\|\vec{v}_{2\perp}\|} = \frac{1}{\sqrt{(-1)^2+1^2+0^2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{v}_{3\parallel} = (\hat{u}_1 \cdot \vec{v}_3)\,\hat{u}_1 + (\hat{u}_2 \cdot \vec{v}_3)\,\hat{u}_2 = \frac{1}{\sqrt{3}}\,(-1+0+1)\,\hat{u}_1 + \frac{1}{\sqrt{2}}\,(1+0+0)\,\hat{u}_2 = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{v}_{3\perp} = \vec{v}_3 - \vec{v}_{3\parallel} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

$$\hat{u}_3 = \frac{\vec{v}_{3\perp}}{\|\vec{v}_{3\perp}\|} = \frac{1}{\sqrt{\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2+1^2}} \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

$$\hat{u}_1 = \tfrac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \hat{u}_2 = \tfrac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \hat{u}_3 = \tfrac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$