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Assignment 3
3.2.25

Which of these functionals on $\mathcal{C}[0,1]$ are linear functionals?

(a) $\int_0^1 t\,|p(t)|\,dt$
(b) $\max_{0\leq t\leq 1} p(t)$
(c) $\int_0^1 p^2(t)dt$
(d) $\int_0^1 p(t)\sin^2 t\,dt$

A function $A(t)$ is linear if

$$A(k\overrightarrow{u} + \overrightarrow{v}) = kA(\overrightarrow{u}) + A(\overrightarrow{v})$$

for all $\overrightarrow{u}$ and $\overrightarrow{v}$ in the domain of A when k is a real constant.
Therefore, we will test the functionals given above according to this rule in order to determine which, if any, are linear.

(a) $\int_0^1 t\,|p(t)|\,dt$

Set $A(p(t)) = \int_0^1 t\,|p(t)|\,dt$.
Then
$A(ku(t) + v(t)) = \int_0^1 t\,|ku(t) + v(t)|\,dt$
The absolute value of a sum is not the sum of the absolute values, as can be shown by considering the absolute value of 6+-8 as opposed to the absolute value of 6 plus the absolute value of -8. Therefore,$|ku(t) + v(t)|$ is not equal to $|ku(t)| + |v(t)|$, and the functional is not linear.

(b)$A(p(t)) = \max_{0\leq t\leq 1} p(t)$
$A(ku(t) + v(t) = \max_{0\leq t\leq 1}(ku(t) + v(t))$
Here, the max function, which returns the maximum value of a function, will result in this functional not being linear. This is because the maximum value of the sum of two functions on a given interval will not be necessarily equal to the sum of the maximum values of the two functions (unless the functions have their maxima at the same point). Consider $y = x^2$ and $y = -x$ for an example. On the interval [0,1], the maxima of the two functions are (1,1) and (0,0) respectively. Therefore, the sum of the two maxima is 0+1=1. But the maxima of the function $y = x^2 - x$ occur at (0,0) and (1,0), and therefore the maximum value on the interval is 0, which does not equal 1.

Therefore, this functional is not linear.

(c) $A(p(t)) = \int_0^1 p^2(t)dt$
$A(ku(t) + v(t)) = \int_0^1 (ku(t) + v(t))^2 dt$
As with the others, the square of the sum does not necessarily equal the sum of the squares (to see this, examine 5 and 6; 25+36 does not equal 121), and so this functional is not linear.

(d) $A(p(t)) = \int_0^1 p(t) \sin^2 t \, dt$
$A(ku(t) + v(t)) = \int_0^1 (ku(t) + v(t)) \sin^2 t \, dt$
$A(ku(t) + v(t)) = \int_0^1 ((ku(t) \sin^2 t) + (v(t) \sin^2 t))dt$
$A(ku(t) + v(t)) = \int_0^1 ku(t) \sin^2(t) \, dt \; + \int_0^1 v(t) \sin^2(t) \, dt$
$A(ku(t) + v(t)) = k \int_0^1 u(t) \sin^2(t) \, dt \; + \int_0^1 v(t) \sin^2(t) \, dt$
$A(ku(t) + v(t)) = kA(u(t)) + A(v(t))$

This functional is linear.