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Assignment 6
4.3.1
http://people.tamu.edu/~kmt4088/ch4sec3prob1.pdf

Proposition 1:
(a) If a set $\mathcal{S}$ is independent, then any subset of $\mathcal{S}$ is also independent.
(b) If a set $\mathcal{S}$ spans $\mathcal{V}$, then any superset of $\mathcal{S}$ (i.e., a set of which $\mathcal{S}$ is a subset) also spans $\mathcal{V}$.

Proof:

(a) The definition of independence for a set is (from page 154):

The set $\mathcal{S} \equiv \{\overrightarrow{v_1}, ..., \overrightarrow{v_k}\}$ is independent iff every $\overrightarrow{y}$ in the vector space can be expressed in at most one way as a linear combination of the vectors $\overrightarrow{v_j}$.

In other words, any given vector in the space under consideration can be expressed in no ways or in one way as a linear combination of the vectors of the set.

Consider the subset $\mathcal{S}_{\text{subset}} \equiv \{\overrightarrow{v_1}, ..., \overrightarrow{v_h}\}$, where $h < k$. Given that $\mathcal{S}$ is independent, consider a vector $\overrightarrow{y}$ in the vector space. There are three possibilities:

(1) $\overrightarrow{y}$ cannot be expressed as a linear combination of the vectors in $\mathcal{S}$. ($\mathcal{S}$ does not span the vector space.) If this is the case, $\overrightarrow{y}$ also cannot be expressed as a linear combination of the vectors in $\mathcal{S}_{\text{subset}}$, because a subset of a set cannot include vectors that were not in the original set.

(2) $\overrightarrow{y}$ can be expressed as a linear combination of the vectors in $\mathcal{S}$ in exactly one way. However, the subset $\mathcal{S}_{\text{subset}}$ does not include one or more of the vectors needed to express $\overrightarrow{y}$. Therefore, there is no way to express $\overrightarrow{y}$ as a linear combination of the vectors in $\mathcal{S}_{\text{subset}}$.

(3) $\overrightarrow{y}$ can be expressed as a linear combination of the vectors in $\mathcal{S}$ in exactly one way. The subset $\mathcal{S}_{\text{subset}}$ includes all the needed vectors, so $\overrightarrow{y}$ can be expressed as a linear combination of the vectors in $\mathcal{S}_{\text{subset}}$ in exactly one way (as mentioned before, a subset cannot include vectors that were not part of the original set; so the caveat "exactly one way" holds).

From these three possibilities, we see that any given vector $\overrightarrow{y}$ in the vector space can be expressed in either no ways or in one way as a linear combination of the vectors in a subset of $\mathcal{S}$. Therefore, when $\mathcal{S}$ is independent, any subset of $\mathcal{S}$ is also independent.

(b) The definition of span for a set is (from page 151):

A set $\mathcal{S} \equiv \{\overrightarrow{v_1}, ..., \overrightarrow{v_k}\}$ in the vector space $\mathcal{V}$ spans $\mathcal{V}$ iff every $\overrightarrow{y}$ in $\mathcal{V}$ can be expressed in at least one way as a linear combination of the vectors $\overrightarrow{v_j}$.

Consider the superset $\mathcal{S}_{\text{superset}} \equiv \{\overrightarrow{v_1}, ..., \overrightarrow{v_m}\}$, where $m > k$. Given that $\mathcal{S}$ spans the vector space $\mathcal{V}$, consider a vector $\overrightarrow{y}$ in $\mathcal{V}$. The definition above shows that $\overrightarrow{y}$ can be expressed in at least one way as a linear combination of the vectors in $\mathcal{S}$. The superset $\mathcal{S}_{\text{superset}}$ contains the vectors in $\mathcal{S}$, so $\overrightarrow{y}$ can also be expressed as a linear combination of the vectors in $\mathcal{S}_{\text{superset}}$. There is a possibility that the additional vectors in $\mathcal{S}_{\text{superset}}$ provide more ways to express $\overrightarrow{y}$, but since the definition places no limit on the number of ways, this is not a concern. Therefore, when a set $\mathcal{S}$ spans a vector space $\mathcal{V}$, any superset of $\mathcal{S}$ must also span $\mathcal{V}$.