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Assignment 8
5.2.12
http://people.tamu.edu/~kmt4088/ch5sec2prob12.pdf

Define a function $G : \mathcal{C}(-\infty,\infty) \to \mathcal{C}(-\infty,\infty)$ by

$[Gf](x) \equiv \int_0^x tf(t)dt$.

(a) Prove that G is linear.
(b) Show that G is injective.
(c) Show that G is *not* surjective. (In other words, find an element of $\mathcal{C}(-\infty,\infty)$ that is not equal to Gf for any f.)

(a) To show that G is linear, consider $G(au(x)+v(x)$, where $a \in \mathbf{R}$ and $u(x), v(x) \in \mathcal{C}(-\infty,\infty)$. Then

$G(au(x)+v(x)) = \int_0^x t(au(t)+v(t))dt$
$G(au(x)+v(x)) = \int_0^x tau(t)dt + \int_0^x tv(t)dt$
$G(au(x)+v(x)) = a\int_0^x tu(t)dt + \int_0^x tv(t)dt$
$G(au(x)+v(x)) = a[Gu](x) + [Gv](x)$

Therefore, G is linear.

(b) G is injective if $[Gu](x) = [Gv](x)$ implies $u(x) = v(x)$.

$[Gu](x) = [Gv](x)$
$\int_0^x tu(t)dt = \int_0^x tv(t)dt$

If the integrals are the same, the derviatives will be the same. Therefore,

$xu(x) = xv(x)$
$u(x) = v(x)$

Therefore, G is injective.

(c) G is surjective if the range of $G : \mathcal{C}(-\infty,\infty) \to \mathcal{C}(-\infty,\infty)$ is $\mathcal{C}(-\infty,\infty)$. Therefore, if an element of $\mathcal{C}(-\infty,\infty)$ can be found that is not equal to Gf for any f, then the range of G is not all of $\mathcal{C}(-\infty,\infty)$.

Consider $[Gu](x) = 5$. This is certainly a continuous function.

$xu(x) = 5$
$u(x) = \frac{5}{x}$

Since $u(x)$ is not continuous, it cannot be used as an argument for G; G is supposed to map elements of $\mathcal{C}(-\infty,\infty)$. Therefore, for the constant functions, which are elements of $\mathcal{C}(-\infty,\infty)$, no f exists that is an element of $\mathcal{C}(-\infty,\infty)$, as required by the definition of G. Therefore, G is not surjective; it cannot map onto the constant functions.

Another way to look at this is to consider $[Gf](0)$. $\int_0^0 tf(t)dt = 0$ for all t, so any function Gf in the range satisfies $[Gf](0) = 0$. Many functions, however, violate this condition.