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Assignment 9
6.1.3 - Revised
http://people.tamu.edu/~kmt4088/ch6sec1prob3.pdf

Prove that the norm

$\|\overrightarrow{x}\| \equiv \sqrt{\langle \overrightarrow{x}, \overrightarrow{x} \rangle}$

defined by any inner product does indeed satisfy the triangle inequality, N3.

Let us begin by considering the Cauchy-Schwarz inequality, which states:

(a) $|\langle \overrightarrow{x}, \overrightarrow{y} \rangle| \leq \|\overrightarrow{x}\| \, \|\overrightarrow{y}\|$

By the definition of the norm stated in the problem, consider

(b) $\langle \overrightarrow{x}, \overrightarrow{y} \rangle \leq \sqrt{\langle \overrightarrow{x}, \overrightarrow{x} \rangle} \sqrt{\langle \overrightarrow{y}, \overrightarrow{y} \rangle}$

Now, consider RI3, the property of bilinearity. It states:

$\langle \overrightarrow{x} + \overrightarrow{y}, \overrightarrow{x} + \overrightarrow{y} \rangle = \langle \overrightarrow{x}, \overrightarrow{x} \rangle + \langle \overrightarrow{y}, \overrightarrow{y} \rangle + 2 \langle \overrightarrow{x}, \overrightarrow{y} \rangle$

Therefore, perform algebraic operations on (b):

(c) $2 \langle \overrightarrow{x}, \overrightarrow{y} \rangle \leq 2 \sqrt{\langle \overrightarrow{x}, \overrightarrow{x} \rangle} \sqrt{\langle \overrightarrow{y}, \overrightarrow{y} \rangle}$
(d) $\langle \overrightarrow{x}, \overrightarrow{x} \rangle + \langle \overrightarrow{y}, \overrightarrow{y} \rangle + 2 \langle \overrightarrow{x}, \overrightarrow{y} \rangle \leq \langle \overrightarrow{x}, \overrightarrow{x} \rangle + \langle y, \overrightarrow{y} \rangle + 2 \sqrt{\langle \overrightarrow{x}, \overrightarrow{x} \rangle} \sqrt{\langle \overrightarrow{y}, \overrightarrow{y} \rangle}$

Then, by RI3, we obtain the inequality:

(e) $\langle \overrightarrow{x} + \overrightarrow{y}, \overrightarrow{x} + \overrightarrow{y} \rangle \leq \langle \overrightarrow{x}, \overrightarrow{x} \rangle + \langle y, \overrightarrow{y} \rangle + 2 \sqrt{\langle \overrightarrow{x}, \overrightarrow{x} \rangle} \sqrt{\langle \overrightarrow{y}, \overrightarrow{y} \rangle}$

Take the square root of each side to obtain

(f) $\sqrt{\langle \overrightarrow{x} + \overrightarrow{y}, \overrightarrow{x} + \overrightarrow{y} \rangle} \leq \sqrt{\langle \overrightarrow{x}, \overrightarrow{x} \rangle} + \sqrt{\langle \overrightarrow{y}, \overrightarrow{y} \rangle}$

which, by the definition of the norm given above, equals

(g) $\|\overrightarrow{x} + \overrightarrow{y}\| \leq \|\overrightarrow{x}\| + \|\overrightarrow{y}\|$.
This is the triangle inequality, N3.

Therefore, the norm defined by any inner product satisfies the triangle inequality.