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Assignment 10
6.2.4
http://people.tamu.edu/~kmt4088/ch6sec2prob4.pdf

Define an inner product on the space of polynomials by

$\langle f, g\rangle = \int_0^2 e^t f(t)g(t)dt.$

Find the first three orthogonal polynomials with respect to this inner product. (Apply the Gram-Schmidt process to the standard basis, $\{1, t, t^2, \ldots\}$.) Suggestion: Start by finding a formula or making a table for the integrals $\int_0^2 e^t t^n dt$ for at least the first few values of n.

I will make the suggested table for values of n from 0 to 3. The first is fairly straightforward:

$\int_0^2 e^t t^0 dt = \int_0^2 e^t dt = e^2 - 1$

But the rest require integration by parts. I will work out the first in full, with explanation, then merely show the work for the next two.

The integration by parts rule says that $\int u dv = uv - \int v du$. Therefore, when integrating a function of the form $e^t t^n$ with respect to t, we must choose one factor as u and the other as dv. The logical choice for dv is $e^t dt$, because then v will be simply e^t. (Note that since we are evaluating a definite integral, we need not consider constants of integration.) Thus, u will be t^n and du will be $nt^{n-1}dt$.

$\int_0^2 e^t t^1 dt = te^t\big|_0^2 - \int_0^2 e^t(1t^0)dt$
$\int_0^2 te^t dt = te^t\big|_0^2 - \int_0^2 e^t dt$
$\int_0^2 te^t dt = 2e^2 - 0e^0 - e^2 + e^0$
$\int_0^2 te^t dt = e^2 + 1$

Similarly,

$\int_0^2 t^2 e^t dt = t^2 e^t\big|_0^2 - 2\int_0^2 te^t dt$

But since we have already found $\int_0^2 te^t dt = e^2 + 1$,

$\int_0^2 t^2 e^t dt = 4e^2 - 0 - 2e^2 - 2 = 2e^2 - 2.$

And finally,

$\int_0^2 t^3 e^t dt = t^3 e^t\big|_0^2 - 3\int_0^2 t^2 e^t dt$
$\int_0^2 t^3 e^t dt = 8e^2 - 0 - 6e^2 + 6 = 2e^2 + 6.$

So, the completed table is

n	$\int_0^2 t^n e^t dt$
0	$e^2 - 1$
1	$e^2 + 1$
2	$2e^2 - 2$
3	$2e^2 + 6$

Now, let us begin applying the Gram-Schmidt method shown in the text to finding the first three orthogonal polynomials (note that we will actually be finding the first two *orthonormal* polynomials, and then a third orthogonal polynomial). Let us set $v_k = t^k$, with the resulting polynomials denoted by $\overrightarrow{u_k}$.

$\|v_0\|^2 = \int_0^2 e^t dt = e^2 - 1$

Now, normalize this vector to obtain

$\hat{u_0} = \sqrt{e^2 - 1}$

and consider

$\langle \hat{u_0}, v_1 \rangle = \int_0^2 e^t \sqrt{(e^2-1)} t dt$

Since $\sqrt{(e^2-1)}$ is a constant, pull it out of the integral to leave $\langle \hat{u_0}, v_1 \rangle = \sqrt{(e^2-1)} \int_0^2 te^t dt$. Refer to the table above to find that $\int_0^2 te^t dt = e^2 + 1$.

$\langle \hat{u_0}, v_1 \rangle = \sqrt{(e^2-1)}(e^2+1) = (e^2-1)^{\frac{3}{2}}$
$v_{1\|} = \langle \hat{u_0}, v_1 \rangle \, \hat{u_0} = (e^2-1)^{\frac{3}{2}}(e^2-1)^{\frac{1}{2}} = (e^2-1)^2$

Then

$v_{1\perp} = v_1 - v_{1\|} = t - (e^2-1)^2$

and

$\hat{u_1} = (1 + (e^2-1)^4)^{-\frac{1}{2}}(t - (e^2-1)^2)$.

To find the third orthogonal polynomial, continue the process. Consider $\langle \hat{u_0}, v_2 \rangle$ and $\langle \hat{u_1}, v_2 \rangle$.

$\langle \hat{u_0}, v_2 \rangle = \int_0^2 e^t \sqrt{(e^2-1)} t^2 dt = \sqrt{(e^2-1)} \int_0^2 t^2 e^t dt = \sqrt{(e^2-1)}(2e^2-2) = 2(e^2-1)^{\frac{3}{2}}$

The second is slightly more complicated.

$\langle \hat{u_1}, v_2 \rangle = \int_0^2 e^t (1 + (e^2-1)^4)^{-\frac{1}{2}}(t - (e^2-1)^2 t^2 dt$
$\langle \hat{u_1}, v_2 \rangle = (1 + (e^2-1)^4)^{-\frac{1}{2}} \left[\int_0^2 t^3 e^t dt - (e^2-1)^2 \int_0^2 t^2 e^t dt \right]$
$\langle \hat{u_1}, v_2 \rangle = (1 + (e^2-1)^4)^{-\frac{1}{2}} [(2e^2+6) - (e^2-1)^2(2e^2-2)] = 2(1 + (e^2-1)^4)^{-\frac{1}{2}}(e^2 + 3 - (e^2-1)^3)$

Now

$v_{2\|} = \langle \hat{u_0}, v_2 \rangle \, \hat{u_0} + \langle \hat{u_1}, v_2 \rangle \, \hat{u_1}$
$v_{2\|} = 2(e^2-1)^{\frac{3}{2}}(e^2-1)^{\frac{1}{2}} + 2(1 + (e^2-1)^4)^{-\frac{1}{2}}(e^2 + 3 - (e^2-1)^3)(1 + (e^2-1)^4)^{-\frac{1}{2}}(t - (e^2-1)^2)$

$v_{2\perp} = v_2 - v_{2\|} = t^2 - 2(e^2-1)^2 + 2(1 + (e^2-1)^4)^{-1}(e^2 + 3 - (e^2-1)^3)(t - (e^2-1)^2)$, so
$\overrightarrow{u_2} = t^2 + 2(e^2 + 3 - (e^2-1)^3)t - 2(e^2-1)^2(1 + (e^2 + 3 - (e^2-1)^3)(1 + (e^2-1)^4)^{-1})$

Therefore, the first three orthogonal polynomials are

$\hat{u_0} = \sqrt{e^2 - 1}$
$\hat{u_1} = (1 + (e^2-1)^4)^{-\frac{1}{2}}(t - (e^2-1)^2)$
$\overrightarrow{u_2} = t^2 + 2(e^2 + 3 - (e^2-1)^3)t - 2(e^2-1)^2(1 + (e^2 + 3 - (e^2-1)^3)(1 + (e^2-1)^4)^{-1})$