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Assignment 11
7.3.5
http://people.tamu.edu/~kmt4088/ch7sec3prob5.pdf

In $\mathbf{R}^3$ introduce "slanted" cylindrical coordinates (r, θ, h) by

$x = r\cos\theta,$
$y = r\sin\theta,$
$z = h + r\cos\theta.$

(a) Calculate the tangent vectors to the coordinate curves and the normal vectors to the coordinate surfaces.

(b) Calculate the mass of a region bounded by the surfaces $r = 2$, $h = 0$, and $h = 1$, if the mass density is $\rho(r, \theta, h) = 10 + r^2$.

(a) As is perhaps unsurprising given that this section focuses on Jacobi's theorem, answering this problem will require calculating the Jacobian, $T'(\overrightarrow{u})$. Then, the columns of this matrix are the tangent vectors to the coordinate curves, while the rows of the inverse of this matrix are the normal vectors to the coordinate surfaces.

$$T'(\overrightarrow{u}) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial h} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial h} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial h} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ \cos\theta & -r\sin\theta & 1 \end{pmatrix}$$

Thus, the tangent vectors to the coordinate curves are

$(\cos\theta, \sin\theta, \cos\theta)$
$(-r\sin\theta, r\cos\theta, -r\sin\theta)$
$(0, 0, 1)$

Now, we need to find the inverse of $T'(\overrightarrow{u})$, T'^{-1}.

$$\left(\begin{array}{ccc|ccc} \cos\theta & -r\sin\theta & 0 & 1 & 0 & 0 \\ \sin\theta & r\cos\theta & 0 & 0 & 1 & 0 \\ \cos\theta & -r\sin\theta & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} \cos\theta & -r\sin\theta & 0 & 1 & 0 & 0 \\ \sin\theta & r\cos\theta & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & -r\tan\theta & 0 & \frac{1}{\cos\theta} & 0 & 0 \\ \sin\theta & r\cos\theta & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & -r\tan\theta & 0 & \frac{1}{\cos\theta} & 0 & 0 \\ 0 & \frac{r}{\cos\theta} & 0 & -\tan\theta & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

[Note: In obtaining the value $\frac{r}{\cos\theta}$, I made use of the trignometric identity $\sin^2\theta + \cos^2\theta = 1$. Specifically: $r\cos\theta + r\frac{\sin^2\theta}{\cos\theta} = r(\frac{\cos^2\theta}{\cos\theta} + \frac{\sin^2\theta}{\cos\theta}) = r(\frac{1}{\cos\theta})$.]

$$\left(\begin{array}{ccc|ccc} 1 & -r\tan\theta & 0 & \frac{1}{\cos\theta} & 0 & 0 \\ 0 & 1 & 0 & -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 1 & 0 & -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

Therefore, the normal vectors to the coordinate surfaces are

$(\cos\theta, \sin\theta, 0)$
$(-\frac{\sin\theta}{r}, \frac{\cos\theta}{r}, 0)$
$(-1, 0, 1)$

(b) To accomplish this, we will need to evalute the integral $\int\int\int_{\mathbf{R}^3}(10+r^2) \mid \det T' \mid dr d\theta dh$. This integral is the result of applying Jacobi's theorem.

To calculate the determinant of T', I prefer the method of reducing the matrix to a triangular matrix and then multiplying the elements of the diagonal. Care must be taken to observe any changes made to the determinant due to the row operations.

$$\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ \cos\theta & -r\sin\theta & 1 \end{pmatrix} \to \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ 0 & \frac{r}{\cos\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

According to the text, if a scalar multiple of one row is added to another row, the determinant does not change. Since all operations were of this type, no changes to the determinant have occurred, and simply multiply the row elements to obtain the determinant.

$\cos\theta \cdot \frac{r}{\cos\theta} \cdot 1 = r$

The integral then becomes

$\int_0^1 \int_0^{2\pi} \int_0^2 (10+r^2) r dr d\theta dh$
$= 2\pi \int_0^2 (10r + r^3) dr = 2\pi[5r^2 + \frac{1}{4}r^4]_0^2 = 2\pi[5\cdot 4 + \frac{1}{4}\cdot 16 - 5\cdot 0 - \frac{1}{4}\cdot 0] = 2\pi[24] = 48\pi$

The mass is 48π units.