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Assignment 12
7.6.9
http://people.tamu.edu/~kmt4088/ch7sec6prob9.pdf

Still in the context of Exercise 7.6.7, calculate the flux $\int\int_S \overrightarrow{A} \cdot d\overrightarrow{S}$ through the "polar cap" $\theta < \pi/3$. (Here θ is the polar angle (colatitude) defined by $z = Rcos\theta$, with $R = 4$ in this case.)

To clarify, Exercise 7.6.7 asks us to consider the sphere with radius 4 centered at the origin in $\mathbf{R}^3$, and the vector field $\overrightarrow{A}(\overrightarrow{r}) = z\hat{k} = (0, 0, z)$. The normal vector points outward.

We will use the parametrization

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4\sin\theta\cos\phi \\ 4\sin\theta\sin\phi \\ 4\cos\theta \end{pmatrix}.$$

To find $d\overrightarrow{S}$, consider page 384, which tells us that

$\int\int_S \overrightarrow{A} \cdot d\overrightarrow{S} = \int\int_{g^{-1}(S)} A_x \frac{\partial(y,z)}{\partial(u,v)} + A_y \frac{\partial(z,x)}{\partial(u,v)} + A_z \frac{\partial(x,y)}{\partial(u,v)} dudv$, where

$\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}$, and $\frac{\partial(z,x)}{\partial(u,v)}$ and $\frac{\partial(x,y)}{\partial(u,v)}$ are defined similarly.

Then, since

$\frac{\partial x}{\partial \theta} = R\cos\theta\cos\phi$, $\frac{\partial y}{\partial \theta} = R\cos\theta\sin\phi$, $\frac{\partial z}{\partial \theta} = -R\sin\theta$, $\frac{\partial x}{\partial \phi} = -R\sin\theta\sin\phi$, $\frac{\partial y}{\partial \phi} = R\sin\theta\cos\phi$, and $\frac{\partial z}{\partial \phi} = 0$,

$\frac{\partial(y,z)}{\partial(\theta,\phi)} = \frac{\partial y}{\partial \theta}\frac{\partial z}{\partial \phi} - \frac{\partial y}{\partial \phi}\frac{\partial z}{\partial \theta} = (R\cos\theta\sin\phi)(0) - (R\sin\theta\cos\phi)(-R\sin\theta) = R^2\sin^2\theta\cos\phi$

and, by similar calculation,

$\frac{\partial(z,x)}{\partial(\theta,\phi)} = R^2\sin^2\theta\sin\phi$
$\frac{\partial(x,y)}{\partial(\theta,\phi} = R^2\cos\theta\sin\theta.$

Then

$d\overrightarrow{S} = R^2[\sin^2\theta\cos\theta\hat{i} + \sin^2\theta\sin\phi\hat{j} + \cos\theta\sin\theta\hat{k}]d\theta d\phi$ (as noted on page 385).

Therefore, we have now to consider

$\int\int_S \overrightarrow{A} \cdot d\overrightarrow{S} = \int\int_S (z\hat{k} \cdot 4^2[\sin^2\theta\cos\theta\hat{i} + \sin^2\theta\sin\phi\hat{j} + \cos\theta\sin\theta\hat{k}]d\theta d\phi.$

This reduces, since $\hat{k}\cdot\hat{i} = \hat{k}\cdot\hat{j} = 0$ and $\hat{k}\cdot\hat{k} = 1$, to

$\int\int_S \overrightarrow{A} \cdot d\overrightarrow{S} = 16\int\int_S z\cos\theta\sin\theta d\theta d\phi.$

Now, since $z = 4\cos\theta$, and we are integrating over all of ϕ from 0 to 2π and over θ from 0 to $\pi/3$,

$\int\int_S \overrightarrow{A} \cdot d\overrightarrow{S} = 64\int_0^{2\pi}\int_0^{\pi/3}\cos^2\theta\sin\theta d\theta d\phi$

Since the integrand is not a function of ϕ,

$\int\int_S \overrightarrow{A} \cdot d\overrightarrow{S} = 128\pi\int_0^{\pi/3}\cos^2\theta\sin\theta d\theta d\phi.$

By the rule of integration that states

$\int \sin cx\cos^n cx dx = -\frac{1}{c(n+1)}\cos^{n+1} cx,$

$\int\int_S \overrightarrow{A} \cdot d\overrightarrow{S} = 128\pi\left[-\frac{1}{1(2+1)}\cos^3\theta\right]_0^{\pi/3}$. Therefore,

$\int\int_s \overrightarrow{A} \cdot d\overrightarrow{S} = 128\pi\cdot -\frac{1}{3}\left(\left(\frac{1}{2}\right)^3 - 1^3\right) = \frac{112\pi}{3}.$