Raymond Waniska (r-waniska@calclab.math.tamu.edu)

Assignment 6, Problem 4.4.12

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http://calclab.math.tamu.edu/~r-waniska/44C.pdf

Problem 4.4.12:

Show that $\{b_1, \ldots\}$ is a basis (for R^4) and find the coordinates of the vector $x = \begin{pmatrix} 2 \\ 4 \\ 2 \\ 4 \end{pmatrix}$ in that basis.

$$\vec{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \ \vec{b}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \ \vec{b}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \ \vec{b}_4 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Solution:

To show that these are a basis, we need to show that they are linearly independent. By putting them in a matrix, we can use row reduction to find this out.

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \\ -1 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

Since the last row is not all zeros, the vectors are a basis for R^4. Then $\vec{x}$ can be expressed as a combination of that basis, so we can write:

$$A\vec{b}_1 + B\vec{b}_2 + C\vec{b}_3 + D\vec{b}_4 = \vec{x}$$

$$\begin{pmatrix} 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 4 \\ 1 & -1 & 1 & 1 & 2 \\ -1 & 1 & 1 & 1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & -1 & 2 \\ 0 & 2 & 0 & 2 & 8 \\ 0 & 0 & 2 & 2 & 6 \\ -1 & 1 & 1 & 1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & -1 & 2 \\ 0 & 2 & 0 & 2 & 8 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 2 & 2 & 0 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 0 & 3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 1 & -2 & -2 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -3 & -5 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & -2 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -3 & -5 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

So $\vec{x}$ expressed in terms of the other vectors

$$\vec{x} = \vec{b}_1 + 2\vec{b}_2 + \vec{b}_3 + 2\vec{b}_4.$$