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Assignment 8, Problem 5.3.1
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http://calclab.math.tamu.edu/˜r-waniska/531.pdf

(a) Find the kernel of $L : C^2(-\infty,\infty) \to C(-\infty,\infty)$defined by $L(u) = u'' - 4u$.

(b) What can you say about the range of L? Show that the range contains (at least) all bounded continuous functions. HINT: Construct the solution by the method of variation of parameters. The point of this problem is that almost every function is in the range, if we put no conditions on the behavior of the solution at infinity. We assume boundedness to guarantee that the integrals converge.

(a) The kernel is the input to the linear function that results in a zero vector output. The general solutions, c_1e^{2t} and c_2e^{-2t} can be found using the methods from a differential equations course.

(b) Try

$$u(t) = A(t)e^{2t} + B(t)e^{-2t}$$

as a solution to

$$u'' - 4u = f(t)$$

$$u'(t) = 2A(t)e^{2t} - 2B(t)e^{-2t}$$

In the method of variation of parameters, we assume $A'(t)e^{2t} + B'(t)e^{-2t} = 0$, and then we differntiate again.

$$u''(t) = 4A(t)e^{2t} + 4B(t)e^{-2t} + 2A'(t)e^{2t} - 2B'(t)e^{-2t}$$

Plugging these into the original equation, we get

$$4A(t)e^{2t} + 4B(t)e^{-2t} + 2A'(t)e^{2t} - 2B'(t)e^{-2t} - 4A(t)e^{2t} - 4B(t)e^{-2t} = f(t)$$

$$2A'(t)e^{2t} - 2B'(t)e^{-2t} = f(t)$$

Now, using the assumption and this equation, we can solve for $A'(t)$ and $B'(t)$, and then integrate to find $A(t)$ and $B(t)$.

$$A'(t)e^{2t} + B'(t)e^{-2t} = 0$$

$$A'(t)e^{2t} = -B'(t)e^{-2t}$$

$$2A'(t)e^{2t} - 2B'(t)e^{-2t} = f(t)$$

$$2(-B'(t)e^{-2t}) - 2B'(t)e^{-2t} = f(t)$$

$$-4B'(t)e^{-2t} = f(t)$$

$$B'(t)e^{-2t} = -\frac{1}{4}f(t),\ A'(t)e^{2t} = \frac{1}{4}f(t)$$

$$B'(t) = -\frac{1}{4}e^{2t}f(t),\ A'(t) = \frac{1}{4}e^{-2t}f(t)$$

$$B(t) = -\frac{1}{4}\int_{-\infty}^{+\infty} e^{2t}f(t)dt,\ A(t) = \frac{1}{4}\int_{-\infty}^{+\infty} e^{-2t}f(t)dt$$

Since there exists a function for all $B(t)$ and $A(t)$, the range is the entire space, $L(u)$ is onto.