White, David d-white1
Assignment 6 4.2.6
REVISED
http://people.tamu.edu/~dmw1435/4_2_6.dvi
http://people.tamu.edu/~dmw1435/4_2_6.pdf

For 4.2.4 - 4.2.6, the coordinates (u, v) are defined by $x = u^2$ and $y = u + v$, for $u > 0$, in the region of $\mathbf{R}^2$ where $x > 0$.

Problem 4.2.6:

Sketch a few of the coordinate curves in the (x, y) plane. Sketch the two sets of vectors at two points, one near the line $x = 0$ and one farther away. Comment on what happens as x approaches 0 and as x approaches infinity.

Note:

The "two sets of vectors" come from the previous problems, 4.2.4 and 4.2.5. Problem 4.2.4 asks for the tangent vectors to the coordinate curves. Problem 4.2.5 asks for the normal vectors to the coordinate curves.

In order to sketch the coordinate curves, we can set u and v constant individually and look at the resulting functions.

First, set v to a constant, say $v = 0$. We obtain $x = u^2$ and $y = u$. This is a parabola opening towards the x-axis. Since the coordinate definitions only allow $u > 0$, we only concern ourselves with the upper half of any such parabolas. Any other constant for v will shift this half of a parabola up or down, for positive or negative v's, respectively.

Next, set u to a constant, say $u = 1$. This gives us $x = 1$ and $y = 1 + v$. This is a vertical line at $x = u^2$.

Figure 1 shows an example of some coordinate curves.

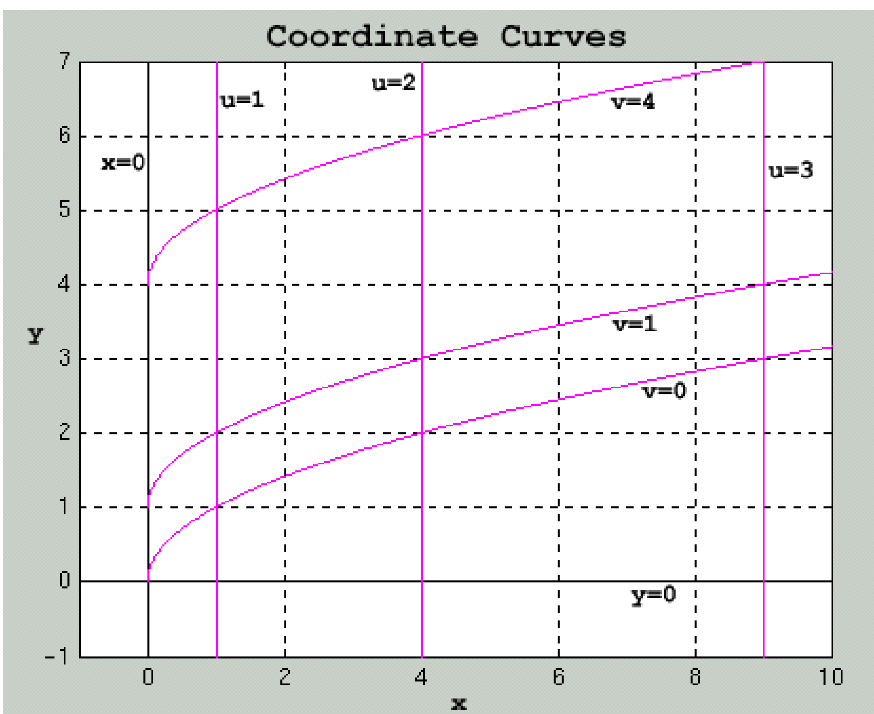

Figure 1 - (u,v) Coordinate Curves Plotted on (x,y) Plane

Now, we need to find the two sets of vectors.

Problem 4.2.4:

To find the tangent vectors, we differentiate x and y with respect to u and v.

$$x = u^2 \qquad\qquad y = u + v$$

$$\frac{\partial}{\partial u}x = 2u \qquad \frac{\partial}{\partial u}y = 1 \qquad \frac{\partial}{\partial u}(x,y) = (2u,1)$$

$$\frac{\partial}{\partial v}x = 0 \qquad \frac{\partial}{\partial v}y = 1 \qquad \frac{\partial}{\partial v}(x,y) = (0,1)$$

Problem 4.2.5:

To find the normal vectors, take the gradients of u and v.

$$v = y - u$$

$$u = \sqrt{x} \qquad\qquad v = y - \sqrt{x}$$

$$\nabla u = (\tfrac{\partial}{\partial x}u, \tfrac{\partial}{\partial y}u) = (\tfrac{1}{2\sqrt{x}}, 0) = (\tfrac{1}{2u}, 0)$$

$$\nabla v = (\tfrac{\partial}{\partial x}v, \tfrac{\partial}{\partial y}v) = (-\tfrac{1}{2\sqrt{x}}, 1) = (-\tfrac{1}{2u}, 1)$$

Back to Problem 4.2.6:

Two points, at which to plot the above sets of vectors, were arbitrarily chosen: $(u,v) = (\frac{1}{2}, 1)$ and $(u,v) = (2,2)$. The corresponding (x,y) coordinates are $(\frac{1}{4}, \frac{3}{2})$ and $(4,4)$.

For $(u,v) = (\frac{1}{2}, 1)$,

$\frac{\partial}{\partial u}(x,y) = (2u,1) = (1,1)$, $\frac{\partial}{\partial v}(x,y) = (0,1)$, and

$\nabla u = (\frac{1}{2u}, 0) = (1,0)$, $\nabla v = (-\frac{1}{2u}, 1) = (-1,1)$.

For $(u,v) = (2,2)$,

$\frac{\partial}{\partial u}(x,y) = (2u,1) = (4,1)$, $\frac{\partial}{\partial v}(x,y) = (0,1)$, and

$\nabla u = (\frac{1}{2u}, 0) = (\frac{1}{4}, 0)$, $\nabla v = (-\frac{1}{2u}, 1) = (-\frac{1}{4}, 1)$.

Figure 2 shows these vectors plotted at their points on a graph of the coordinate curves.

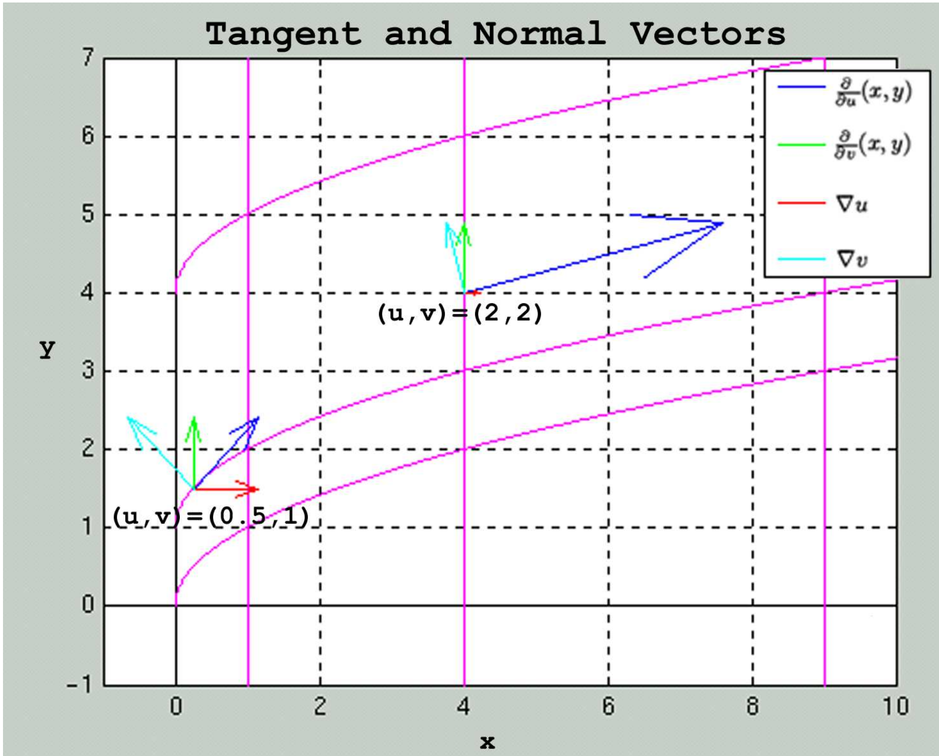

Figure 2 - Tangent and Normal Vectors Plotted on Coordinate Curves

When $x \to \infty$, or similarly $u \to \infty$, the tangent vector basis becomes

$$\begin{aligned} \frac{\partial}{\partial u}(x,y) &= (\infty, 1) \\ \frac{\partial}{\partial v}(x,y) &= (0,1), \end{aligned}$$

and the normal vector basis becomes

$$\begin{aligned} \nabla u &= (0,0) \\ \nabla v &= (0,1). \end{aligned}$$

When $x \to 0$, or $u \to 0$, the tangent vector basis becomes

$$\begin{aligned} \frac{\partial}{\partial u}(x,y) &= (0,1) \\ \frac{\partial}{\partial v}(x,y) &= (0,1), \end{aligned}$$

and the normal vector basis becomes

$$\begin{aligned} \nabla u &= (\infty, 0) \\ \nabla v &= (-\infty, 1). \end{aligned}$$

In each of these cases, the dimension of the span of these bases is 1 at the limits. A one-dimensional span cannot be a basis for $\mathbf{R}^2$.