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Assignment 7 4.5.15

http://people.tamu.edu/~dmw1435/4_5_15.dvi
http://people.tamu.edu/~dmw1435/4_5_15.pdf

Find the matrix of L with respect to the unprimed basis for $\mathbf{R}^4$ given in Exercise 4.4.19 if its matrix with respect to the primed basis in that exercise is

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

From 4.4.19,

$$\vec{b}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{b}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{b}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{ and}$$

$$\vec{b'}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{b'}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \vec{b'}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{b'}_4 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Let's define the matrix of L with respect to the primed basis as M. If we define the argument coordinates of the primed basis for $\mathbf{R}^4$ as $\vec{r}$, then the problem gives us the relationship

$$L(\vec{r}) = M\vec{r}. \tag{1}$$

We can also define the argument coordinates of the unprimed basis for $\mathbf{R}^4$ as $\vec{s}$. The relationship we are looking for, then, is

$$L(\vec{s}) = N\vec{s}. \tag{2}$$

To start solving this change of basis problem, we can write out a coordinate mapping of $\vec{b}$ into $\vec{b'}$.

$$\begin{pmatrix} b'_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Since the above was a coordinate mapping, we can write,

$$\vec{r} = A\vec{s} \tag{3}$$

$$\vec{s} = A^{-1}\vec{r}, \tag{4}$$

and since the basis of the operator is consistent, we can also write,

$$L(\vec{r}) = AL(\vec{s}). \tag{5}$$

$$L(\vec{s}) = A^{-1}L(\vec{r}). \tag{6}$$

Now, we simply substitute Equation (6) into Equation (2), then substitute Equation (1) for $L(\vec{r})$, and finally Equation (3) for $\vec{r}$.

$$\begin{aligned}
L(\vec{s}) &= N\vec{s} \\
A^{-1}L(\vec{r}) &= N\vec{s} \\
A^{-1}M\vec{r} &= N\vec{s} \\
A^{-1}MA\vec{s} &= N\vec{s} \\
\therefore A^{-1}MA &= N
\end{aligned}$$

$$\begin{aligned}
N &= A^{-1}MA = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\
N &= \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}
\end{aligned}$$