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Assignment 8 5.3.1

http://people.tamu.edu/˜dmw1435/5_3_1.dvi
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(a) Find the kernel of $L : \mathcal{C}^2(-\infty,\infty) \to \mathcal{C}(-\infty,\infty)$ defined by $L(u) = u" - 4u$.

(b) What can you say about the range of L? Show that the range contains (at least) all bounded continuous functions. HINT: Construct the solution by the method of variation of parameters.The point of this problem is that almost every function is in the range, if we put no conditions on the behavior of the solution at infinity. We assume boundedness to guarantee that the integrals converge.

Part (a)

The kernel is the set of all solutions to the homogeneous function. We can find homogeneous solutions by the auxilliary equation $r^2 - 4 = 0$. This equation yields

$$\begin{aligned} r^2 - 4 &= 0 \\ r^2 &= 4 \\ r &= \pm 2 \end{aligned}$$

The homogeneous solution is of the form

$$u_h = c_1 e^{2t} + c_2 e^{-2t}.$$

Therefore, the kernel is the set of all linear combinations of e^{2t} and e^{-2t}.

Part (b)

The range is the set of all elements that $L(u)$ could possibly contain. We need to show that the range contains all bounded continuous functions by variation of parameters. If the range contains a bounded function, say $cos(t)$, this means $L(u) = u" - 4u = cos(t)$. We need to show that a function u satisfies this equation in order for $cos(t)$ to be in the range.

Variation of parameters:

$$M = \begin{pmatrix} e^{2t} & e^{-2t} \\ 2e & -2e^{-2t} \end{pmatrix} \text{ (form Wronskian matrix of } u_h\text{), where } u_h = \begin{pmatrix} e^{2t} & e^{-2t} \end{pmatrix}$$
$$\vec{b} = \begin{pmatrix} 0 \\ cos(t) \end{pmatrix} \text{ (form column vector of 'solution')}$$

$$\begin{aligned} M\vec{v'} &= \vec{b} \\ \vec{v'} &= M \backslash \vec{b} \\ \vec{v'} &= M^{-1}\vec{b} \end{aligned}$$

Then,

$$\begin{aligned} \vec{v} &= \int \vec{v'} \\ \vec{v} &= \begin{pmatrix} -\frac{1}{10}cos(t)e^{-2t} + \frac{1}{20}sin(t)e^{-2t} \\ -\frac{1}{10}cos(t)e^{2t} - \frac{1}{20}sin(t)e^{2t} \end{pmatrix} \end{aligned}$$

The particular solution is given by $u_p = u_h \vec{v}$. Thus,

$$u_p = -\frac{1}{5}cos(t).$$

The above calculations were made using MATLAB. It was also checked that if we set $L(u) = sin(t)$ instead of $cos(t)$, we arrive at the particular solution $u_p = -\frac{1}{5}sin(t)$. We know that $cos(t)$ and $sin(t)$ are bounded and from the above, that they are in the range of L. Another note to make is that the range of L must include the set of all 2^{nd} derivatives of twice-differentiable functions.