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Assignment 9 | 5.5.10

http://people.tamu.edu/~dmw1435/5_5_10.dvi
http://people.tamu.edu/~dmw1435/5_5_10.pdf

Again suppose that $x^2y + yz = 0$ and $xyz + 1 = 0$, but this time regard x and y as functions of z. Find the equation of the tangent line to the curve at any point where $z = -2$. How many such points are there?

Since we are given the equations

$$\begin{array}{rcl} x^2y + yz & = & 0 \\ xyz + 1 & = & 0 \end{array},$$

we can form the matrix

$$G(\vec{x}, \vec{y}) = 0 = \begin{pmatrix} x^2y + yz \\ xyz + 1 \end{pmatrix}.$$

In this problem, the independent variable $\vec{x} = z$, and the dependent variable $\vec{y} = x, y$. As shown in the text (pg 261),

$$\frac{d\vec{y}}{d\vec{x}} = -\left(\frac{\nabla\vec{G}}{\nabla\vec{y}}\right)^{-1} \frac{\nabla\vec{G}}{\nabla\vec{x}} = \begin{pmatrix} \frac{dx}{dz} \\ \frac{dy}{dz} \end{pmatrix}, \quad (1)$$

where

$$\frac{\nabla\vec{G}}{\nabla\vec{y}} = \begin{pmatrix} \frac{\nabla G_1}{\nabla x} & \frac{\nabla G_1}{\nabla y} \\ \frac{\nabla G_2}{\nabla x} & \frac{\nabla G_2}{\nabla y} \end{pmatrix} = \begin{pmatrix} 2xy & x^2 + z \\ yz & xz \end{pmatrix}, \text{ and}$$

$$\frac{\nabla\vec{G}}{\nabla\vec{x}} = \begin{pmatrix} \frac{\nabla G_1}{\nabla z} \\ \frac{\nabla G_2}{\nabla z} \end{pmatrix} = \begin{pmatrix} y \\ yx \end{pmatrix}.$$

We can obtain $\frac{d\vec{y}}{d\vec{x}}$ by plugging these into Equation 1:

$$\begin{array}{rcl} \frac{d\vec{y}}{d\vec{x}} = \begin{pmatrix} \frac{dx}{dz} \\ \frac{dy}{dz} \end{pmatrix} & = & -\begin{pmatrix} 2xy & x^2 + z \\ yz & xz \end{pmatrix}^{-1} \begin{pmatrix} y \\ yx \end{pmatrix} \\ & = & -\frac{1}{x^2 - z} \begin{pmatrix} \frac{x}{y} & -(x^2 + z) \\ -1 & 2x \end{pmatrix} \begin{pmatrix} y \\ yx \end{pmatrix} \\ & = & \frac{1}{x^2 - z} \begin{pmatrix} -x + \frac{x(x^2+z)}{z} \\ y - \frac{2x^2y}{z} \end{pmatrix}. \end{array}$$

Since we are looking for the tangent at the points where $z = -2$, we can substitute -2 for z in the above result.

$$\frac{d\vec{y}}{d\vec{x}} = \begin{pmatrix} \frac{dx}{dz} \\ \frac{dy}{dz} \end{pmatrix} = \frac{1}{x^2 + 2} \begin{pmatrix} -x - \frac{1}{2}x(x^2 - 2) \\ y + x^2y \end{pmatrix}.$$

An equation for the tangent line to a curve (giving y in terms of x) where $z = -2$ is given by

$$\vec{y} = \vec{y}_0 + \frac{dy}{dx}(x - \vec{x}_0).$$

In order to find $\frac{dy}{dx}$, we divide $\frac{dy}{dz}$ by $\frac{dx}{dz}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dz}}{\frac{dx}{dz}} = \frac{y + x^2y}{-x - \frac{1}{2}x(x^2 - 2)} \left(\frac{2}{2}\right) = \frac{2y + 2x^2y}{-2x - x(x^2 - 2)}.$$

At this point, we need to step back and find the x and $y$coordinates associated with a z value of -2.

If $z = -2$, the equations become

$$\begin{aligned} x^2y - 2y = 0 \quad &\text{and} \quad -2xy + 1 = 0 \\ x^2 = 2 \quad & \qquad\quad y = \frac{1}{2x} \\ x = \pm\sqrt{2} \quad & \qquad\quad y = \frac{\pm\sqrt{2}}{4}. \end{aligned}$$

From this, we can see that there are <u>two points</u>: $(\vec{x}_0, \vec{y}_0) = \left\{ \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} \end{pmatrix} \right\}$.

At this point, we should also solve for $\left.\frac{dy}{dx}\right|_{(x_0,y_0)}$.

$$\begin{aligned} \frac{dy}{dx} &= \left.\left(\frac{2y + 2x^2y}{-2x - x(x^2-2)}\right)\right|_{(x_0,y_0)} \\ &= \begin{pmatrix} \left.\left(\frac{2y+2x^2y}{-2x-x(x^2-2)}\right)\right|_{(\sqrt{2},\frac{\sqrt{2}}{4})} \\ \left.\left(\frac{2y+2x^2y}{-2x-x(x^2-2)}\right)\right|_{(-\sqrt{2},-\frac{\sqrt{2}}{4})} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}}}{-2\sqrt{2}} \\ \frac{-\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}}}{2\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} \\ -\frac{3}{4} \end{pmatrix} \end{aligned}$$

For both points, $\frac{dy}{dx} = -\frac{3}{4}$. The final equation for the tangent line (y-coordinates in terms of x-coordinates) to the curve where $z = -2$ is

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} \end{pmatrix} - \frac{3}{4} \begin{pmatrix} x_1 - \sqrt{2} \\ x_2 + \sqrt{2} \end{pmatrix}$$