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Assignment 9 — 6.1.9

http://people.tamu.edu/~dmw1435/6_1_9.dvi
http://people.tamu.edu/~dmw1435/6_1_9.pdf

Use the Cauchy-Schwartz inequality to give an alternative proof of Exercise 3.3.2, working entirely with the usual norm $\|\vec{x}\|$ instead of introducing the alternative norm $|\vec{x}|$.

The Cauchy-Schwartz inequality is $|<\vec{x},\vec{y}>| \leq \|\vec{x}\|\,\|\vec{y}\|$.

Exercise 3.3.2:

Prove that any linear function $f : \mathbf{R}^n \rightarrow \mathbf{R}^p$ is continuous. Recommended steps: show continuity for $\vec{x} = \vec{0}$, derive $\epsilon - \delta$ statement of theorem to be proved, show that $|f(\vec{x})|$ is less than some constant times $|\vec{x}|$.

First, define $f(\vec{x}) = M\vec{x}$. When $\vec{x} = \vec{0}$, $f(\vec{0}) = M\vec{0} = \vec{0}$.

Definition (pg 113), f is continuous at $\vec{x}\epsilon\mathbf{R}^n$ if

$$\lim_{\vec{z}\rightarrow\vec{x}} f(\vec{z}) = f(\vec{x}).$$

$f(\vec{x})$ is continuous at $\vec{x} = \vec{0}$ if

$$\lim_{\vec{z}\rightarrow\vec{0}} f(\vec{z}) = \vec{0},$$

which by the definition of the limit (pg 113) means that for every $\epsilon > 0$ there is a $\delta > 0$ such that $0 < \|\vec{z}\| < \delta$ implies $\|f(\vec{z})\| < \epsilon$.

$$\|f(\vec{z})\| \quad = \|M\vec{z}\| \quad = \sqrt{\sum_{j=1}^{m}\left(\sum_{k=1}^{n} M_{jk}z_k\right)^2} < \epsilon$$

$$\|f(\vec{z})_j\| \quad = \|M_j\vec{z}\| \quad = \sqrt{\left(\sum_{k=1}^{n} M_{jk}z_k\right)^2} < \epsilon$$

$$\|f(\vec{z})_j\| = \sum_{k=1}^{n} M_{jk}z_k \quad =< M, \vec{z} >_j \quad \leq \|M_j\|\,\|\vec{z}\| = \epsilon$$

$$\|f(\vec{z})\| \leq \sum_{j=1}^{m} \|M_j\|\,\|\vec{z}\| = m\,\|M\|\,\|\vec{z}\| = \epsilon$$

Here, we can define $\epsilon = m\,\|M\|\,\delta$, where we can take $\delta \geq \|\vec{z}\|$.

We can write

$$\|f(\vec{z})\| \leq m\,\|M\|\,\delta.$$

Now, we can show that the limit exists (for the proof of continuity at $x = \vec{0}$):

$f(\vec{x})$ is continuous at $\vec{x} = \vec{0}$ if

$$\lim_{\vec{z}\rightarrow\vec{0}} f(\vec{z}) = \vec{0},$$

which means that for every $\epsilon > 0$ there is a $\delta > 0$ such that $0 < \|\vec{z}\| < \delta$ implies $\|f(\vec{z})\| < \epsilon$. We also know that $\|f(\vec{z})\| \leq m\,\|M\|\,\delta = m\,\|M\|\,\|\vec{z}\|$, so the limit exists for any non-infinite $\|\vec{z}\|$. Therefore, $f(\vec{x})$ is continuous at $\vec{x} = \vec{0}$.

Now, we need to check the case where $\vec{x} \neq \vec{0}$. Since f is linear, $f(\vec{z} - \vec{x}) = f(\vec{z}) - f(\vec{x})$.
$f(\vec{x})$ is continuous at $\vec{x} \in \mathbf{R}^n$ if

$$\lim_{\vec{z} \to \vec{x}} f(\vec{z}) = f(\vec{x})$$

$$\Downarrow$$

$$\lim_{\vec{z} \to \vec{x}} f(\vec{z}) - f(\vec{x}) = \vec{0}$$

$$\Downarrow$$

$$\lim_{\vec{z} - \vec{x} \to \vec{0}} f(\vec{z} - \vec{x}) = \vec{0},$$

which means that for every $\epsilon > 0$ there is a $\delta > 0$ such that $0 < \|\vec{z} - \vec{x}\| < \delta$ implies $\|f(\vec{z} - \vec{x})\| < \epsilon$. For the $\vec{x} = \vec{0}$, $\vec{z} \to \vec{0}$ case, we showed that

$$\|f(\vec{z})\| \leq m \|M\| \delta = m \|M\| \|\vec{z}\|.$$

We can say that similarly, we have

$$\|f(\vec{z} - \vec{x})\| \leq m \|M\| \delta = m \|M\| \|\vec{z} - \vec{x}\|.$$

The limit exists. Therefore, $f(\vec{x})$ is continuous at $\vec{x} \in \mathbf{R}^n$. $f(\vec{x})$is continuous for all $\vec{x}$.