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Assignment 12 7.5.4
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Explain the following statement:

The identity $\nabla \cdot (\nabla \times \vec{B}) = 0$ is necessary to prevent an inconsistency between Stokes's theorem and Gauss's theorem.

HINT: Consider two curves that have the same endpoints.

Assume we have a volume integral $\int_V (\nabla \cdot \vec{A}) d^3r$. The volume is contained within a closed surface. Gauss's theorem provides the relation $\int_V (\nabla \cdot \vec{A}) d^3r = \oint_S \vec{A} \cdot d\vec{S}$. The volume integral has the same value as the closed-surface integral. Let's define a curve C to exist on the closed surface. We can now divide the closed surface into two surfaces that each have C as a boundary. Similarly, the value of the closed-surface integral will be equal to the sum of integrals of the surfaces, $\oint_S \vec{A} \cdot d\vec{S} = \int_{S1} \vec{A} \cdot d\vec{S} + \int_{S2} \vec{A} \cdot d\vec{S}$.

Now, Stokes's theorem relates the values of the integral over the closed curve C and the integral over the surface it surrounds, $\oint_C \vec{B} \cdot d\vec{r} = \int_S (\nabla \times \vec{B}) \cdot d\vec{S}$. We can now write the relationship between the line integral and the volume integral.

$$\int_V (\nabla \cdot \vec{A}) d^3r = \oint_S \vec{A} \cdot d\vec{S} = \int_{S1} \vec{A} \cdot d\vec{S} + \int_{S2} \vec{A} \cdot d\vec{S} = \int_{S1} (\nabla \times \vec{B}) \cdot d\vec{S} + \int_{S2} (\nabla \times \vec{B}) \cdot d\vec{S} = \oint_{C1} \vec{B} \cdot d\vec{r} + \oint_{C2} \vec{B} \cdot d\vec{r}.$$

However, because C is a boundary to the two surfaces that contain a volume, the normal vectors to the surfaces will point in generally opposite directions and, therefore, the path around C will be taken in different directions by each surface's line integral. This tells us that $C1 = -C2$. Now, the relationship is

$$\int_V (\nabla \cdot (\nabla \times \vec{B})) d^3r = \oint_S (\nabla \times \vec{B}) \cdot d\vec{S} = \int_{S1} (\nabla \times \vec{B}) \cdot d\vec{S} + \int_{S2} (\nabla \times \vec{B}) \cdot d\vec{S} = \oint_{C1} \vec{B} \cdot d\vec{r} - \oint_{C1} \vec{B} \cdot d\vec{r} = 0, \text{ or simply}$$

$$\int_V (\nabla \cdot (\nabla \times \vec{B})) d^3r = 0.$$

At the outset of the problem, we said that there *is* a volume within the closed surface, so $\int_V d^3r \neq 0$. This tells us that $\nabla \cdot (\nabla \times \vec{B}) = 0$.