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Assignment 13 — 8.1.13

http://people.tamu.edu/~dmw1435/8_1_13.dvi
http://people.tamu.edu/~dmw1435/8_1_13.pdf

Assume that A is a diagonalizable matrix (independent of t) and that all its eigenvalues are positive. Show that the solution of the second-order homogeneous linear differential equation

$$\frac{d^2\vec{x}}{dt^2} = -A\vec{x}; \qquad \vec{x}(0) = \vec{x}_0, \qquad \vec{x}'(0) \equiv \left.\frac{d\vec{x}}{dt}\right|_{t=0} = \vec{v}_0,$$

can be written in terms of trigonometric functions of the matrix $t\sqrt{A}$, acting on the vectors $\vec{x}_0$ and $\vec{v}_0$.

Since A is diagonalizable, we can find its eigenvalues, and eigenvectors, to form matrix U, which contains the eigenvectors in its columns, and matrix D, which has the eigenvalues along its diagonal. By the second theorem on page 397 of the text, we can write the following relationship between A, D, and U,

$$\begin{aligned} A &= UDU^{-1}, \text{ and} \\ D &= U^{-1}AU. \end{aligned}$$

Matrix U maps eigenvector coordinates to natural basis coordinates, while U^{-1} maps natural coordinates into eigen-coordinates.

To keep things relatively simple, we can write that $\vec{w} = U^{-1}\vec{x}$, where $\vec{w}$ is the eigenspace version of $\vec{x}$.We rewrite the differential:

$$\begin{aligned} \frac{d^2\vec{x}}{dt^2} &= -A\vec{x} \\ \frac{d^2\vec{w}}{dt^2} = U^{-1}\frac{d^2\vec{x}}{dt^2} &= -U^{-1}A\vec{x} \\ \frac{d^2\vec{w}}{dt^2} &= -U^{-1}AU\vec{w} \\ \frac{d^2\vec{w}}{dt^2} &= -D\vec{w} = -\begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\vec{w}. \end{aligned}$$

After having studied differential equations for a semester, we know that the solution to $w_1''(t) = -\lambda_1 w(t)$ consists of cosine and sine parts, $w_1(t) = \cos(t\sqrt{\lambda_1})c_1 + \sin(t\sqrt{\lambda_1})c_2$. We can write $\vec{w}(t)$ as

$$\begin{aligned} \vec{w}(t) &= \begin{pmatrix} \cos(t\sqrt{\lambda_1}) & 0 & \cdots \\ 0 & \cos(t\sqrt{\lambda_2}) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} c_1 + \begin{pmatrix} \sin(t\sqrt{\lambda_1}) & 0 & \cdots \\ 0 & \sin(t\sqrt{\lambda_2}) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} c_2 \\ \vec{w}(t) &= \cos(t\sqrt{D})c_1 + \sin(t\sqrt{D})c_2 \\ \vec{w}(t) &= \begin{pmatrix} \cos(t\sqrt{D}) & \sin(t\sqrt{D}) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \end{aligned}$$

Since the elements of $\vec{c}$ are constants in the eigenspace, we can define natural basis constants $\vec{k}$ by $\vec{c} = U^{-1}\vec{k}$. Using the $\vec{w}$-$\vec{x}$ and A-D relations first introduced, along with the new definition for $\vec{k}$, we can write $\vec{x}(t)$ as follows.

$$\begin{aligned}
\vec{x}(t) &= U\vec{w}(t) = U \begin{pmatrix} \cos(t\sqrt{D}) & \sin(t\sqrt{D}) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
\vec{x}(t) &= U \begin{pmatrix} \cos(t\sqrt{D}) & \sin(t\sqrt{D}) \end{pmatrix} U^{-1} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \\
\vec{x}(t) &= \begin{pmatrix} \cos(t\sqrt{A}) & \sin(t\sqrt{A}) \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.
\end{aligned}$$

Now, we can examine initial conditions.

$$\begin{aligned}
\vec{x}(0) = \vec{x}_0 &= \begin{pmatrix} \cos(0\sqrt{A}) & \sin(0\sqrt{A}) \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \\
\vec{x}_0 &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = k_1 \\
k_1 &= \vec{x}_0.
\end{aligned}$$

$$\begin{aligned}
\frac{d\vec{x}(t)}{dt} &= \begin{pmatrix} -\sqrt{A}\sin(t\sqrt{A}) & \sqrt{A}\cos(t\sqrt{A}) \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \\
\frac{d\vec{x}(0)}{dt} = \vec{v}_0 &= \begin{pmatrix} -\sqrt{A}\sin(0\sqrt{A}) & \sqrt{A}\cos(0\sqrt{A}) \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \\
\vec{v}_0 &= \begin{pmatrix} 0 & \sqrt{A} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \sqrt{A}k_2 \\
k_2 &= \frac{\vec{v}_0}{\sqrt{A}}.
\end{aligned}$$

We now know what the constants are for $\vec{x}(t)$, so we can write the complete solution.

$$\vec{x}(t) = \begin{pmatrix} \cos(t\sqrt{A}) & \sin(t\sqrt{A}) \end{pmatrix} \begin{pmatrix} \vec{x}_0 \\ \frac{\vec{v}_0}{\sqrt{A}} \end{pmatrix}$$